Time delay for the Dirac equation.*†

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Abstract

We consider time delay for the Dirac equation. A new method to calculate the asymptotics of the expectation values of the operator $\int_0^\infty e^{iH_0t}\zeta\left(\frac{|x|}{R}\right)e^{-iH_0t}dt$, as $R\to\infty$, is presented. Here H_0 is the free Dirac operator and $\zeta(t)$ is such that $\zeta(t)=1$ for $0\le t\le 1$ and $\zeta(t)=0$ for $t>1$. This approach allows us to obtain the time delay operator $\delta\mathcal{T}(f)$ for initial states f in $\mathcal{H}_2^{3/2+\varepsilon}\left(\mathbb{R}^3;\mathbb{C}^4\right)$, $\varepsilon>0$, the Sobolev space of order $3/2+\varepsilon$ and weight 2. The relation between the time delay operator $\delta\mathcal{T}(f)$ and the Eisenbud-Wigner time delay operator is given. Also, the relation between the averaged time delay and the spectral shift function is presented.

1 Introduction.

In the present paper we consider time delay for the quantum scattering pair $\{H_0,H\}$, where the free Dirac operator H_0 is given by

$$H_0=-i\alpha\cdot\nabla+m\alpha_4, \tag{1.1}$$

with m - the mass of the particle, $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ and α_j, $j=1,2,3,4$, are 4×4 Hermitian matrices that satisfy the relation:

$$\alpha_j\alpha_k+\alpha_k\alpha_j=2\delta_{jk},\ \ j,k=1,2,3,4,$$

where δ_{jk} denotes the Kronecker symbol. The standard choice of α_j is ([33]):

$$\alpha_j=\begin{pmatrix}0&\sigma_j\\ \sigma_j&0\end{pmatrix},\ \ 1\le j\le 3,\qquad \alpha_4=\begin{pmatrix}I_2&0\\0&-I_2\end{pmatrix}=\beta,$$

(I_n is the $n\times n$ unit matrix) and

$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

are the Pauli matrices. The operator H_0 is a self-adjoint operator on $L^2\left(\mathbb{R}^3;\mathbb{C}^4\right)$ (see Section 2). The perturbed Dirac operator is defined by

$$H=H_0+\mathbf{V}. \tag{1.2}$$

Here the potential $\mathbf{V}(x)$ is an Hermitian 4×4 matrix valued function defined for $x\in\mathbb{R}^3$ such that H is a self-adjoint operator on $L^2\left(\mathbb{R}^3;\mathbb{C}^4\right)$ (see Section 2). For a detailed study of the Dirac equation we refer to [8], [33], and the references quoted there.

We define time delay for the pair $\{H_0,H\}$ as follows. Let $\zeta(t)$ be such that $\zeta(t)=1$ for $0\le t\le1$ and $\zeta(t)=0$ for $t>1$. For $R>0$ and a normalized $f\in L^2\left(\mathbb{R}^3;\mathbb{C}^4\right)$ we define the quantities

$$\mathcal{T}_{0,R}(f):=\int_{-\infty}^{\infty}\left\|\zeta\left(\frac{|x|}{R}\right)e^{-iH_0t}f\right\|^2dt=\int_{-\infty}^{\infty}\int_{|x|\le R}\left|e^{-iH_0t}f\right|^2_{\mathbb{C}^4}dt$$

*AMS Subject Classifications: 35Q40; 35P25; 35Q41; 81U99
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and

$$\mathcal{T}_R(f) := \int_{-\infty}^{\infty} \left\| \zeta\left(\frac{|x|}{R}\right) e^{-iHt} f \right\|^2 dt = \int_{-\infty}^{\infty} \int_{|x|\le R} \left| e^{-iHt} f \right|^2_{\mathbb{C}^4} dt.$$

As $\left\| \zeta\left(\frac{|x|}{R}\right) e^{-iH_0 t} f \right\|^2$ is the probability that the state $e^{-iH_0 t} f$ is localized in the ball $B_R := \{x \in \mathbb{R}^3 \big| \, |x| \le R\}$ at time t, $\mathcal{T}_{0,R}(f)$ represents the total time spent in B_R by a normalized state f under the free evolution group $e^{-iH_0 t}$, and similarly, $\mathcal{T}_R(f)$ measures the total time that the state represented by f stays in B_R under e^{-iHt}.

Since the scattering theory is based on the comparison of the perturbed dynamics with the free one, it is natural to define time delay as the difference of the time that the scattered particles stay in the scattering region and the time that the free particles, subject to the same initial conditions, spend in the scattering region. Let f be the initial condition at $t = 0$ that defines the dynamics $e^{-iH_0 t} f$ of the free particle. Then, the wave operator W_-, if it exists, defines the initial condition $W_- f$ in $t = 0$ of the scattered state $e^{-iHt} W_- f$, with the property that asymptotically for $t \to -\infty$ it has the same dynamics as the free state, i.e., $\left\| e^{-iH_0 t} f - e^{-iHt} W_- f \right\| \to 0$, when $t \to -\infty$. We define the time delay in B_R as the difference of the time spent in B_R by the state $W_- f$ and the time that the free state f stays in B_R. That is, the *local time delay* $\delta_R \mathcal{T}(f)$ is given by

$$\delta_R \mathcal{T}(f) := \mathcal{T}_R(W_- f) - \mathcal{T}_{0,R}(f).$$

As the effective scattering region is all of $\mathbb{R}^3$, we have to consider the limit of $\delta_R \mathcal{T}(f)$, when $R \to \infty$. If the limit exists, this leads us to the definition of the *global time delay* or simply *time delay*

$$\delta \mathcal{T}(f) := \lim_{R\to\infty} \delta_R \mathcal{T}(f). \tag{1.3}$$

The main problem in this definition of time delay consists in exhibiting a set of initial states $f \in L^2(\mathbb{R}^3; \mathbb{C}^4)$ and a class of perturbations $H - H_0$ for which the limit in (1.3) exists.

There are a lot of papers concerning time delay for the Schrödinger equation (see [3], [4], [5], [6], [12], [16], [17], [18], [23], [24], [25], [26], [27], [30], [31], [36], and the references therein). The works [17], [23], [24], [29] and [30] study more general, abstract dynamics, however they apply the obtained results to the case of the Schrödinger equation only. Many physical aspects, as well as applications of time delay are presented in [14]. Time delay for dynamics given by a regular enough pseudodifferential operator of hypoelliptic-type, such as the Schrödinger operator or the square-root Klein-Gordon operator (pseudo-relativistic Schrödinger operator), were treated in [34]. However, the Dirac operator was not considered in [34]. As far as we know there are no papers concerning time delay for the Dirac equation.

In order to present our results we make some definitions. For $1 \le p < \infty$ and $\alpha \in \mathbb{R}$ we denote by $L^p(\mathbb{R}^3; \mathbb{C}^4)$ and $\mathcal{H}^\alpha(\mathbb{R}^3; \mathbb{C}^4)$ the Lebesgue and Sobolev spaces of $\mathbb{C}^4$-vector valued functions, respectively (see, for example, [1]). We often will write L^p and $\mathcal{H}^\alpha$ instead of $L^p(\mathbb{R}^3; \mathbb{C}^4)$ and $\mathcal{H}^\alpha(\mathbb{R}^3; \mathbb{C}^4)$. Also, we introduce the weighted L^2 spaces for $s \in \mathbb{R}$, $L^2_s := \{f : \langle x\rangle^s f(x) \in L^2\}$, $\|f\|_{L^2_s} := \|\langle x\rangle^s f(x)\|_{L^2}$, where $\langle x\rangle = \left(1 + |x|^2\right)^{1/2}$. Moreover, for any $\alpha, s \in \mathbb{R}$ we define $\mathcal{H}^\alpha_s := \{f : \langle x\rangle^s f(x) \in \mathcal{H}^\alpha\}$, $\|f\|_{\mathcal{H}^\alpha_s} := \|\langle x\rangle^s f(x)\|_{\mathcal{H}^\alpha}$, where $\|f(x)\|_{\mathcal{H}^\alpha} = \left(\int_{\mathbb{R}^3} \langle\xi\rangle^{2\alpha} \left|\hat{f}(\xi)\right|^2 d\xi\right)^{1/2}$. The Fourier transform $\mathcal{F}$ is given by

$$\hat{f}(\xi) = (\mathcal{F}f)(\xi) := (2\pi)^{-3/2} \int_{\mathbb{R}^3} e^{-i\xi\cdot x} f(x)\, dx.$$

We denote by $(\cdot,\cdot)$ the L^2 scalar product. The scalar product in $\mathbb{C}^4$ is denoted by $\langle\cdot,\cdot\rangle$. The projections $\mathbf{P}_\pm$ on the positive and negative energies of H_0 are defined by (see Section 2)

$$\mathbf{P}_\pm := \frac{1}{2}\left(I_4 \pm \frac{H_0}{|H_0|}\right).$$

Also, we introduce

$$\mathbf{A}_0 := \frac{1}{2i}\left(\frac{1}{|\nabla|^2}\nabla\cdot x + x\cdot\nabla\frac{1}{|\nabla|^2}\right) = \frac{1}{2}\left(\frac{1}{|\nabla|^2}\mathbf{A} + \mathbf{A}\frac{1}{|\nabla|^2}\right), \tag{1.4}$$

where ∇ is the gradient and the operator $\mathbf{A}$, known as "the generator of dilations", is defined as

$$\mathbf{A} := \frac{1}{2i}\sum_{j=1}^{3}(x\cdot\nabla + \nabla\cdot x).$$

The operators $\Gamma_0(E)$, given by (2.4), define the spectral representation $\mathcal{F}_0$ of the operator H_0 (2.5). We denote by $\hat{\mathcal{H}}$ the space of the spectral realization of H_0 under $\mathcal{F}_0$ (see (2.6)) and $(\cdot,\cdot)_{\hat{\mathcal{H}}}$ denotes the scalar product in $\hat{\mathcal{H}}$. Also for any open set $O \subset (-\infty,-m)\cup(m,+\infty)$, we define

$$\Phi(O) := \{ f \in \mathcal{H}_2^{3/2+\varepsilon}\left(\mathbb{R}^3;\mathbb{C}^4\right),\ \varepsilon>0 \Big| \quad f = \psi_f(H_0) f, \quad \text{for some } \psi_f \text{ such that } \operatorname{supp}\psi_f \subset O\}. \tag{1.5}$$

Finally, $\rho(H)$ is the resolvent set of H and $\sigma_p(H)$ is the closure of the set of the eigenvalues of the operator H.

We assume that the wave operators $W_\pm$ exist and are complete, and hence, the scattering operator $\mathbf{S}$ is unitary. In Section 2 we give sufficient conditions on the potential $\mathbf{V}$ under which this assumption is true.

We are now in position to present our results.

Theorem 1.1 *Suppose that the state $f \in \mathcal{H}_2^{3/2+\varepsilon}\left(\mathbb{R}^3;\mathbb{C}^4\right)$, $\varepsilon>0$, is such that $\mathbf{S}f \in \mathcal{H}_2^{3/2+\varepsilon}\left(\mathbb{R}^3;\mathbb{C}^4\right)$, the function $t \to \left\|\left(W_- - e^{itH}e^{-itH_0}\right) f\right\|_{L^2(\mathbb{R}^3)}$ belongs to $L^1((-\infty,0])$ and $t \to \left\|\left(W_+ - e^{itH}e^{-itH_0}\right)\mathbf{S}f\right\|_{L^2(\mathbb{R}^3)}$ belongs to $L^1([0,\infty))$. Then the limit in (1.3) exists and*

$$\delta\mathcal{T}(f) = (f, \mathbf{T}f), \tag{1.6}$$

where

$$\mathbf{T} := H_0\mathbf{S}^*\left(\mathbf{P}_-\left[\mathbf{A}_0,\mathbf{S}\right]\mathbf{P}_- - \mathbf{P}_+\left[\mathbf{A}_0,\mathbf{S}\right]\mathbf{P}_+\right).$$

Moreover, let the scattering matrix $S(E)$ be continuously differentiable with respect to E on some open set $O \subset (-\infty,-m)\cup(m,+\infty)\setminus\sigma_p(H)$. Then, for any $f \in \Phi(O)$, $\mathbf{T}$ is the Eisenbud-Wigner time delay operator, that is

$$\delta\mathcal{T}(f) = \left(\Gamma_0(E) f, T(E)\Gamma_0(E) f\right)_{\hat{\mathcal{H}}},$$

with

$$T(E) := -iS(E)^*\frac{d}{dE}S(E). \tag{1.7}$$

Remark 1.2 Observe that Theorem 1.1 remains valid in the case when $H-H_0$ is not a multiplication operator.

Remark 1.3 We note that in the case of the Schrödinger equation a similar result was proved in [3]. For pseudodifferential operators of hypoelliptic-type, a result as in Theorem 1.1 was obtained in [34]. In these papers, the Fourier transforms of the functions f and Sf are assumed to be compactly supported. We do not need this condition and we prove Theorem 1.1 for f and Sf belonging to the weighted Sobolev space $H_2^{3/2+\varepsilon}\left(\mathbb{R}^3;\mathbb{C}^4\right)$.

We suppose now that the potential $\mathbf{V}$ satisfies the following:

Condition 1.4 The potential $\mathbf{V}$ has the form

$$\mathbf{V}(x) = \langle x\rangle^{-\rho}\left(\mathbf{V}_1(x) + \mathbf{V}_2(x)\right), \quad \rho>2,$$

where each element of the matrix $\mathbf{V}_1$ belongs to $L^\infty\left(\mathbb{R}^3\right)$ and the entries of $\mathbf{V}_2$ are $L^p\left(\mathbb{R}^3\right)$ functions, for some $p>3$. Moreover, the elements of the matrix $x\cdot\nabla\mathbf{V}_1$ are in $L^\infty\left(\mathbb{R}^3\right)+L^{q_1}\left(\mathbb{R}^3\right)$ and the entries of $\langle x\rangle\mathbf{V}_2$ belong to $L^\infty\left(\mathbb{R}^3\right)+L^{q_2}\left(\mathbb{R}^3\right)$, with $q_j \geq 2$, $j=1,2$.

For $\tau>0$ we define the following dense subset of $L^2\left(\mathbb{R}^3;\mathbb{C}^4\right)$:

$$\mathcal{D}_\tau := \{ f \in L^2_\tau\left(\mathbb{R}^3;\mathbb{C}^4\right) \big| \quad f = \psi_f(H_0) f, \quad \text{for some } \psi_f \in C_0^\infty((-\infty,-m)\cup(m,+\infty)\setminus\sigma_p(H))\}.$$

We have

Theorem 1.5 *Suppose that $\mathbf{V}$ satisfies Condition 1.4 and let $f \in \mathcal{D}_\tau$, $\tau>2$. Then, the global time delay $\delta\mathcal{T}(f)$ exists and relation (1.6) is true, with $\mathbf{T}$ being the Eisenbud-Wigner time delay operator.*

Let us now suppose that the potential $\mathbf{V}$ satisfy the following:

Condition 1.6 The potential $\mathbf{V}$ satisfies the estimate

$$|\mathbf{V}(x)| \leq C(1+|x|)^{-4-\varepsilon}, \quad \varepsilon>0.$$

The *spectral shift function* (SSF) is a real valued function $\xi(E;H,H_0)$ such that the relation

$$\operatorname{Tr}(f(H)-f(H_0))=\int_{-\infty}^{\infty}\xi(E;H,H_0)\,f'(E)\,dE,$$

known as the trace formula, holds at least for all $f\in C_0^\infty(\mathbb{R})$ (see [38], [41] and the references there in). Here $\operatorname{Tr}A$ denotes the trace of an operator A. The *average time delay* at energy E is defined by $\operatorname{Tr}T(E)$, with $T(E)$ given by (1.7). Finally, we present a formula that relates the average time delay with the SSF. We have the following result (see [13], [30], [38], [41], and the references therein in the case of the Schrödinger operator).

Theorem 1.7 *Assume that* $\mathbf{V}$ *satisfies Condition 1.6. Then, the following equality is valid*

$$\operatorname{Tr}T(E)=-2\pi\xi'(E;H,H_0), \tag{1.8}$$

for $E\in(-\infty,-m)\cup(m,+\infty)$.

Remark 1.8 For an operator A of trace class we denote by $\operatorname{Det}(I+A)$ the determinant of $I+A$ ([38], [41]). If Condition 1.6 holds, Theorem 4.5 of [40] implies that the operator $S(E)-I$ is of trace class. The scattering phase $\theta(E)$ is defined by the relation $\operatorname{Det}S(E)=e^{-2i\theta(E)}$. It also follows from Theorem 4.5 of [40] that the SSF $\xi(E;H,H_0)$ exists and it is related to the scattering phase $\theta(E)$ by $\xi(E;H,H_0)=(1/\pi)\,\theta(E)$. Thus, formula (1.8) shows that up to numerical coefficients, the average time delay, the SSF and the scattering phase, that have different physical meanings, coincide. On the other hand, we observe that Theorem 1.5 and Theorem 1.7 establish, via the Eisenbud-Wigner operator, a connection between $\delta\mathcal{T}(f)$ and the SSF $\xi(E;H,H_0)$.

We now briefly explain our strategy. As in the case of the Schrödinger equation ([3]), by Proposition 4.1 the study of the limit (1.3) reduces to finding an asymptotic expansion for the quantity

$$I(R):=\int_0^\infty\left\langle e^{-iH_0t}f,\zeta\left(\frac{|x|}{R}\right)e^{-iH_0t}g\right\rangle dt, \tag{1.9}$$

as $R\to\infty$. In order to find the asymptotics of $I(R)$ we proceed as follows. Separating H_0 into positive and negative energies we obtain the decomposition (3.1) below. Since $\sqrt{|p|^2+m^2}+\sqrt{|q|^2+m^2}$ is different from 0 for all p and q, the terms $I_2(R)$ and $I_4(R)$ in (3.1), containing $e^{-i\left(\sqrt{|p|^2+m^2}+\sqrt{|q|^2+m^2}\right)t}$ or $e^{i\left(\sqrt{|p|^2+m^2}+\sqrt{|q|^2+m^2}\right)t}$, result to be $o(1)$ functions, as $R\to\infty$. On the other hand, as $\sqrt{|p|^2+m^2}=\sqrt{|q|^2+m^2}$ for $p=q$, the terms $I_1(R)$ and $I_3(R)$, that contain $e^{-i\left(\sqrt{|p|^2+m^2}-\sqrt{|q|^2+m^2}\right)t}$ or $e^{i\left(\sqrt{|p|^2+m^2}-\sqrt{|q|^2+m^2}\right)t}$, give us the principal part in the asymptotics of $I(R)$, as $R\to\infty$. The asymptotic expansion of $I(R)$ is given by Theorem 3.2. To prove this theorem we first separate the part in $I_1(R)$ and $I_3(R)$ that diverges as R, when $R\to\infty$, and the constant part. These are the results of Lemmas 3.3 and 3.5, respectively. After that, we need to show that the remaining part is $o(1)$ function, as $R\to\infty$. We prove this result in Lemmas 3.4 and 3.6. Finally, in Lemma 3.7 we show that the terms $I_2(R)$ and $I_4(R)$ also are $o(1)$ functions, as $R\to\infty$. This completes the scheme of the proof of Theorem 3.2. We observe that the difficulty in obtaining the asymptotics of $I(R)$ consists in that the integral in t in (1.9) is conditionally convergent and this convergence depends on R. The dependence on R is rather delicate and therefore we need some sharp estimates in weighted Sobolev spaces in order to obtain our result. Also, we note that we do not use a formula analogous to the Alsholm-Kato formula (see (2.1) of [7]) that was used in [3] or [34]. Besides, our approach allows us to obtain the asymptotics of $I(R)$ for functions f,g in weighted Sobolev space $\mathcal{H}_2^{3/2+\varepsilon}$, $\varepsilon>0$, and we do not need that the Fourier transforms of f and g have compact support. This enables us to prove Theorem 1.1 for f and $\mathbf{S}f$ belonging to $\mathcal{H}_2^{3/2+\varepsilon}(\mathbb{R}^3;\mathbb{C}^4)$ (see Remark 1.3). If we assume that f and g in Theorem 3.2 are such that their Fourier transforms are compactly supported, the proof of Theorem 3.2 results to be technically easier. The first assertion of Theorem 1.1 is proved by using Proposition 4.1 and Theorem 3.2, and then, we give the relation of the time delay $\mathbf{T}$ and the Eisenbud-Wigner time delay operator ([15], [37]), which is the result of the second assertion of Theorem 1.1 (see Subsection 4.1). We observe that our method is direct and can be applied to another equations in quantum scattering theory, such as, for example, the Schrödinger operator, the Klein-Gordon equation or the Pauli operator. The proof of Theorem 1.5 consists in showing that under Condition 1.4 on the potential $\mathbf{V}$ the assumptions of Theorem 1.1 are valid. We do this by adapting the results of [4] and [18] for the Schrödinger operator to the case of the Dirac operator. Finally, we prove Theorem 1.7 by using the Birman-Krein's formula, obtained for the Dirac equation in [40].

The paper is organized as follows. In Section 2 we give some known results about scattering theory for the Dirac operator. In Section 3 we obtain the asymptotic expansion for $I(R)$, as $R\to\infty$. Section 4 is dedicated to the proofs of our theorems. In Subsection 4.1 we use the asymptotics of $I(R)$ in order to prove Theorem 1.1. Subsection 4.2 is dedicated to Theorem 1.5. Finally, the proof of Theorem 1.7 is given in Subsection 4.3.

2 Basic notions.

The free Dirac operator H_0 (1.1) is a self-adjoint operator on $L^2(\mathbb{R}^3;\mathbb{C}^4)$ with domain $D(H_0) = \mathcal{H}^1(\mathbb{R}^3;\mathbb{C}^4)$ ([33]). We can diagonalize H_0 by the Fourier transform $\mathcal{F}$. Actually, $\mathcal{F}H_0\mathcal{F}^*$ acts as multiplication by the matrix $h_0(\xi) = \alpha\cdot\xi + m\beta$. This matrix has two eigenvalues $E = \pm\sqrt{\xi^2+m^2}$ and each eigenspace $X^{\pm}(\xi)$ is a two-dimensional subspace of $\mathbb{C}^4$. The orthogonal projections onto these eigenspaces are given by (see [33], page 9)

$$P_{\pm}(\xi) := \frac{1}{2}\left(I_4 \pm \left(\xi^2+m^2\right)^{-1/2}(\alpha\cdot\xi + m\beta)\right). \tag{2.1}$$

Note that

$$P_{\pm}(\xi)\mathcal{F} = (\mathcal{F}\mathbf{P}_{\pm})(\xi),$$

where

$$\mathbf{P}_{\pm} := \frac{1}{2}\left(I_4 \pm \frac{H_0}{|H_0|}\right).$$

The spectrum of H_0 is purely absolutely continuous and it is given by $\sigma(H_0) = \sigma_{ac}(H_0) = (-\infty,-m]\cup[m,\infty)$.

Let us now consider the perturbed Dirac operator H, given by (1.2). Suppose that the Hermitian 4×4 matrix valued potential $\mathbf{V}$, defined for $x\in\mathbb{R}^3$ satisfies the following

Condition 2.1 For some $s_0 > 1/2$, $\langle x\rangle^{2s_0}\mathbf{V}$ is a compact operator from $\mathcal{H}^1$ to L^2.

The assumptions on a potential $\mathbf{V}$, assuring Condition 2.1 are well known (see, for example, [32]). In particular, Condition 2.1 for $\mathbf{V}$ holds, if for some $\varepsilon > 0$, $\sup_{x\in\mathbb{R}^3}\int_{|x-y|\le1}\left|\langle y\rangle^{2s_0}\mathbf{V}(y)\right|^{3+\varepsilon}dy < \infty$ and $\int_{|x-y|\le1}\left|\langle y\rangle^{2s_0}\mathbf{V}(y)\right|^{3+\varepsilon}dy \to 0$, as $|x|\to\infty$ (see Theorem 9.6, Chapter 6, of [32]). Of course, the last two relations are true if $\mathbf{V}$ satisfies Condition 1.4.

Since $\mathbf{V}$ is an Hermitian 4×4 matrix valued potential $\mathbf{V}$, Condition 2.1 implies assumptions (A$_1$)-(A$_3$) of [9]. Thus, under Condition 2.1 H is a self-adjoint operator on $D(H) = \mathcal{H}^1$ and the essential spectrum $\sigma_{ess}(H) = \sigma(H_0)$. The wave operators (WO), defined as the following strong limit

$$W_{\pm}(H,H_0) := s - \lim_{t\to\pm\infty} e^{iHt}e^{-iH_0t},$$

exist and are complete, i.e., $\operatorname{Range} W_{\pm} = \mathcal{H}_{ac}$ (the subspace of absolutely continuity of H) and the singular continuous spectrum of H is absent.

Remark 2.2 We recall that the study about the absence of eigenvalues of H embedded in the absolutely continuous spectrum was made in [10], [19], [35], [42], and the references quoted there. In particular, there are no eigenvalues in the absolutely continuous spectrum if

$$|\mathbf{V}(x)| \le C(1+|x|)^{-1-\varepsilon}, \quad \varepsilon > 0.$$

From the existence of the WO it follows that $HW_{\pm} = W_{\pm}H_0$ (intertwining relations). The scattering operator, defined by

$$\mathbf{S} = \mathbf{S}(H,H_0) := W_+^*W_-,$$

commutes with H_0 and it is unitary.

Let $H_{0S} := -\triangle$ be the free Schrödinger operator in $L^2(\mathbb{R}^3;\mathbb{C}^4)$. The limiting absorption principle (LAP) is the following statement. For z in the resolvent set of H_{0S} let $R_{0S}(z) := (H_{0S}-z)^{-1}$ be the resolvent. The limits $R_{0S}(\lambda\pm i0) = \lim_{\varepsilon\to+0}R_{0S}(\lambda\pm i\varepsilon)$, ($\varepsilon\to+0$ means $\varepsilon\to0$ with $\varepsilon>0$) exist in the uniform operator topology in $\mathcal{B}\left(L_s^2,\mathcal{H}_{-s}^{\alpha}\right)$, $s>1/2$, $|\alpha|\le2$ ([2],[22],[39],[41]) and, moreover, $\|R_{0S}(\lambda\pm i0)f\|_{\mathcal{H}^{\alpha,-s}} \le C_{s,\delta}\lambda^{-(1-|\alpha|)/2}\|f\|_{L_s^2}$, for $\lambda\in[\delta,\infty)$, $\delta>0$. Here for any pair of Banach spaces X,Y, $\mathcal{B}(X,Y)$ denotes the Banach space of all bounded operators from X into Y. The functions $R_{0S}^{\pm}(\lambda)$, given by $R_{0S}(\lambda)$ if $\operatorname{Im}\lambda\neq0$, and $R_{0S}(\lambda\pm i0)$, if $\lambda\in(0,\infty)$, are defined for $\lambda\in\mathbb{C}^{\pm}\cup(0,\infty)$ ($\mathbb{C}^{\pm}$ denotes, respectively, the upper, lower, open complex half-plane) with values in $\mathcal{B}\left(L_s^2,\mathcal{H}_{-s}^{\alpha}\right)$ and they are analytic for $\operatorname{Im}\lambda\neq0$ and locally Hölder continuous for $\lambda\in(0,\infty)$ with exponent ϑ satisfying the estimates $0<\vartheta\le s-1/2$ and $\vartheta<1$.

For z in the resolvent set of H_0 let $R_0(z) := (H_0-z)^{-1}$ be the resolvent. From the LAP for H_{0S} it follows that the limits (see Lemma 3.1 of [9])

$$R_0(E\pm i0) = \lim_{\varepsilon\to+0}R_0(E\pm i\varepsilon) = \begin{cases} (H_0+E)\,R_{0S}\left(\left(E^2-m^2\right)\pm i0\right) & \text{for } E>m \\ (H_0+E)\,R_{0S}\left(\left(E^2-m^2\right)\mp i0\right) & \text{for } E<-m, \end{cases} \tag{2.2}$$

exist for $E\in(-\infty,-m)\cup(m,\infty)$ in the uniform operator topology in $\mathcal{B}\left(L_s^2,\mathcal{H}_{-s}^{\alpha}\right)$, $s>1/2$, $\alpha\le1$, and $\|R_0(E\pm i0)f\|_{\mathcal{H}^{\alpha,-s}}$ $\le C_{s,\delta}|E|^{|\alpha|}\|f\|_{L_s^2}$, for $|E|\in[m+\delta,\infty)$, $\delta>0$. Furthermore, the functions, $R_0^{\pm}(E)$, given by $R_0(E)$, if $\operatorname{Im}E\neq0$, and by

$R_0(E \pm i0)$, if $E \in (-\infty, -m) \cup (m, \infty)$, are defined for $E \in \mathbb{C}^{\pm} \cup (-\infty, -m) \cup (m, +\infty)$ with values in $\mathcal{B}\left(L_s^2, \mathcal{H}_{-s}^{\alpha}\right)$, and moreover, they are analytic for $\operatorname{Im} E \neq 0$ and locally Hölder continuous for $E \in (-\infty, -m) \cup (m, \infty)$ with exponent ϑ such that $0 < \vartheta \leq s - 1/2$ and $\vartheta < 1$.

Next we consider the resolvent $R(z) := (H - z)^{-1}$ for z in the resolvent set of H. The following limits exist for $E \in \{(-\infty, -m) \cup (m, \infty)\} \backslash \sigma_p(H)$ in the uniform operator topology in $\mathcal{B}\left(L_s^2, \mathcal{H}_{-s}^{\alpha}\right)$, $s \in (1/2, s_0]$, $|\alpha| \leq 1$, where s_0 is defined by Condition 2.1 (see Theorem 3.9 of [9])

$$R(E \pm i0) = \lim_{\varepsilon \to +0} R(E \pm i\varepsilon) = R_0(E \pm i0)\left(1 + \mathbf{V} R_0(E \pm i0)\right)^{-1}. \tag{2.3}$$

From this relation and the properties of $R_0^{\pm}(E)$ it follows that the functions, $R^{\pm}(E) := \{R(E)$ if $\operatorname{Im} E \neq 0$, and $R(E \pm i0)$, $E \in \{(-\infty, -m) \cup (m, \infty)\} \backslash \sigma_p(H)\}$, defined for $E \in \mathbb{C}^{\pm} \cup \{(-\infty, -m) \cup (m, \infty)\} \backslash \sigma_p(H)$, with values in $\mathcal{B}\left(L_s^2, \mathcal{H}_{-s}^{\alpha}\right)$ are analytic for $\operatorname{Im} E \neq 0$ and locally Hölder continuous for $E \in \{(-\infty, -m) \cup (m, \infty)\} \backslash \sigma_p(H)$ with exponent ϑ such that $0 < \vartheta \leq s - 1/2$, $s < s_0$ and $\vartheta < 1$.

We now give a spectral representation of H_0. Let us define

$$\left(\Gamma_0(E) f\right)(\omega) := (2\pi)^{-\frac{3}{2}} \upsilon(E) P_{\omega}(E) \int_{\mathbb{R}^3} e^{-i\nu(E)\omega \cdot x} f(x)\, dx, \tag{2.4}$$

where

$$\upsilon(E) = \left(E^2\left(E^2 - m^2\right)\right)^{\frac{1}{4}}, \text{ and } \nu(E) = \sqrt{E^2 - m^2},$$

and

$$P_{\omega}(E) := \begin{cases} P_+(\nu(E)\omega), & E > m, \\ P_-(\nu(E)\omega), & E < -m, \end{cases}$$

($P_{\pm}(\xi)$ are given by (2.1)). The adjoint operator $\Gamma_0^*(E) : L^2\left(\mathbb{S}^2; \mathbb{C}^4\right) \to L_{-s}^2$, $s > 1/2$, is given by

$$\left(\Gamma_0^*(E) f\right)(\omega) := (2\pi)^{-\frac{3}{2}} \upsilon(E) \int_{\mathbb{S}^2} e^{i\nu(E)x \cdot \omega} P_{\omega}(E) f(\omega)\, d\omega.$$

Note that $\Gamma_0(E)$ is unitary equivalent to the trace operator $T_0(E)$, defined by using the Foldy-Wouthuysen transform in [9] (see [28]). Then, from the properties of $T_0(E)$ ([20], [21], [41]) we conclude that $\Gamma_0(E)$ is bounded from L_s^2, $s > 1/2$, into $L^2\left(\mathbb{S}^2; \mathbb{C}^4\right)$ and the operator valued function $\Gamma_0(E)$ is locally Hölder continuous on $(-\infty, -m) \cup (m, \infty)$ with exponent ϑ satisfying $0 < \vartheta \leq s - 1/2$ and $\vartheta < 1$.

Since the operators $\Gamma_0(E)$ and $T_0(E)$ are unitary equivalent, it follows from Section 3 of [9] that the operator

$$\left(\mathcal{F}_0 f\right)(E, \omega) := \left(\Gamma_0(E) f\right)(\omega)) \tag{2.5}$$

extends to unitary operator from L^2 onto

$$\hat{\mathcal{H}} := \int_{(-\infty, -m) \cup (m, +\infty)}^{\oplus} \mathcal{H}(E)\, dE, \tag{2.6}$$

where

$$\mathcal{H}(E) := \int_{\mathbb{S}^2}^{\oplus} X^{\pm}(\nu(E)\omega)\, d\omega, \quad \pm E > m.$$

Moreover, $\mathcal{F}_0$ gives a spectral representation of H_0

$$\mathcal{F}_0 H_0 \mathcal{F}_0^* = E,$$

the operator of multiplication by E in $\hat{\mathcal{H}}$. For these results see [28].

Since the scattering operator $\mathbf{S}$ commutes with H_0, the operator $\mathcal{F}_0 \mathbf{S} \mathcal{F}_0^*$ acts as a multiplication by the operator valued function $S(E) : \mathcal{H}(E) \to \mathcal{H}(E)$. We obtain from Theorem 4.2 of [9] (see also [38],[39],[41]) the following stationary formula for $S(E)$ (see [28]),

$$S(E) = I - 2\pi i \Gamma_0(E)\left(\mathbf{V} - \mathbf{V} R(E + i0) \mathbf{V}\right) \Gamma_0(E)^*, \tag{2.7}$$

for $E \in \{(-\infty, -m) \cup (m, \infty)\} \backslash \sigma_p(H)$. Here I is the identity operator on $\mathcal{H}(E)$. $S(E)$ is called the scattering matrix.

3 Asymptotics for $I(R)$.

Let us first show that $I(R)$ is well defined. The following result holds.

Lemma 3.1 *For any fixed $0<R<\infty$, and $f,g\in\mathcal{H}^{3/2+\varepsilon}_{3/2+\varepsilon}$, $\varepsilon>0$, we have*

$$I(R)<\infty.$$

Proof. Using that $\mathbf{P}_{+}+\mathbf{P}_{-}=I$, we get

$$\begin{aligned} I(R) &= \int_0^\infty \left\langle \mathcal{F}\left(\left(\mathbf{P}_{+}+\mathbf{P}_{-}\right)\left(e^{-iH_0t}f\right)\right), \mathcal{F}\left(\zeta\left(\tfrac{|x|}{R}\right)\left(\mathbf{P}_{+}+\mathbf{P}_{-}\right)e^{-iH_0t}g\right)\right\rangle dt \\ &= (2\pi)^{-3}\int_0^\infty \left\langle \mathcal{F}\left(\left(\mathbf{P}_{+}+\mathbf{P}_{-}\right)e^{-iH_0t}f\right), \tilde{\zeta}_R * \mathcal{F}\left(\left(\mathbf{P}_{+}+\mathbf{P}_{-}\right)e^{-iH_0t}g\right)\right\rangle dt \\ &= (2\pi)^{-3}\left(I_1+I_2+I_3+I_4\right)(R), \end{aligned} \tag{3.1}$$

where $*$ denotes the convolution, $\tilde{\zeta}_R=(2\pi)^{3/2}\mathcal{F}\left(\zeta\left(\frac{|x|}{R}\right)\right)$ and

$$I_1(R):=\int_0^\infty\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\left\langle e^{-i\sqrt{|p|^2+m^2}t}f_{+}(p),\tilde{\zeta}_R(p-q)\,e^{-i\sqrt{|q|^2+m^2}t}g_{+}(q)\right\rangle dqdpdt,$$

$$I_2(R):=\int_0^\infty\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\left\langle e^{-i\sqrt{|p|^2+m^2}t}f_{+}(p),\tilde{\zeta}_R(p-q)\,e^{i\sqrt{|q|^2+m^2}t}g_{-}(q)\right\rangle dqdpdt,$$

$$I_3(R):=\int_0^\infty\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\left\langle e^{i\sqrt{|p|^2+m^2}t}f_{-}(p),\tilde{\zeta}_R(p-q)\,e^{-i\sqrt{|q|^2+m^2}t}g_{+}(q)\right\rangle dqdpdt,$$

$$I_4(R):=\int_0^\infty\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\left\langle e^{i\sqrt{|p|^2+m^2}t}f_{-}(p),\tilde{\zeta}_R(p-q)\,e^{i\sqrt{|q|^2+m^2}t}g_{-}(q)\right\rangle dqdpdt,$$

with $f_{\pm}(p):=P_{\pm}(p)\hat{f}(p)$ and $g_{\pm}(p):=P_{\pm}(p)\hat{g}(p)$. We prove that $|I_1(R)|<\infty$. The proof of $|I_j(R)|<\infty$, for $j=2,3,4$, is similar.

Let us define

$$h(f_{+},g_{+};t):=\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\left\langle e^{-i\sqrt{|p|^2+m^2}t}f_{+}(p),\tilde{\zeta}_R(p-q)\,e^{-i\sqrt{|q|^2+m^2}t}g_{+}(q)\right\rangle dqdp.$$

It is enough to show that

$$\left|\int_0^\infty h(f_{+},g_{+};t)\,dt\right|\le C\left\|f_{+}\right\|_{\mathcal{H}^{3/2+\varepsilon}_{3/2+\varepsilon}(\mathbb{R}^3)}\left\|g_{+}\right\|_{\mathcal{H}^{3/2+\varepsilon}_{3/2+\varepsilon}(\mathbb{R}^3)}. \tag{3.2}$$

Since

$$\left|\tilde{\zeta}_R(p)\right|\le C,\quad p\in\mathbb{R}^3,$$

then

$$\left|\int_0^1 h(f_{+},g_{+};t)\,dt\right|\le C\left\|f_{+}\right\|_{L^1(\mathbb{R}^3)}\left\|g_{+}\right\|_{L^1(\mathbb{R}^3)}\le C\left\|f_{+}\right\|_{L^2_{3/2+\varepsilon}(\mathbb{R}^3)}\left\|g_{+}\right\|_{L^2_{3/2+\varepsilon}(\mathbb{R}^3)},$$

and thus, in order to get (3.2), we need the estimate

$$\left|\int_1^\infty h(f_{+},g_{+};t)\,dt\right|\le C\left\|f_{+}\right\|_{\mathcal{H}^{3/2+\varepsilon}_{3/2+\varepsilon}(\mathbb{R}^3)}\left\|g_{+}\right\|_{\mathcal{H}^{3/2+\varepsilon}_{3/2+\varepsilon}(\mathbb{R}^3)}. \tag{3.3}$$

Suppose that $f,g\in\mathcal{S}$ (here $\mathcal{S}$ denote the Schwartz class). Observe that

$$\begin{aligned} h\left(f_{+},g_{+};t\right) &= \int_{\mathbb{S}^2}\int_{\mathbb{S}^2}\int_0^\infty\int_0^\infty \left\langle e^{-i\sqrt{r^2+m^2}t}f_{+}\left(r\omega\right),\tilde{\zeta}_R\left(r\omega-r_1\omega'\right)e^{-i\sqrt{r_1^2+m^2}t}g_{+}\left(r_1\omega'\right)\right\rangle r^2r_1^2drdr_1d\omega'd\omega \\ &= \frac{1}{t^2}\int_{\mathbb{S}^2}\int_{\mathbb{S}^2}\int_0^\infty\int_0^\infty \left\langle \left(\partial_r e^{-i\sqrt{r^2+m^2}t}\right)\frac{\sqrt{r^2+m^2}}{r}f_{+}\left(r\omega\right),\right. \\ &\quad \times\left.\left(\partial_{r_1}e^{-i\sqrt{r_1^2+m^2}t}\right)\tilde{\zeta}_R\left(r\omega-r_1\omega'\right)\frac{\sqrt{r_1^2+m^2}}{r_1}g_{+}\left(r_1\omega'\right)\right\rangle r^2r_1^2drdr_1d\omega'd\omega. \end{aligned} \tag{3.4}$$

Integrating by parts in both r and r_1, we get

$$\begin{aligned} h\left(f_{+},g_{+};t\right) = \frac{1}{t^2}\int_{\mathbb{S}^2}\int_{\mathbb{S}^2}\int_0^\infty\int_0^\infty & e^{-i\left(\sqrt{r^2+m^2}-\sqrt{r_1^2+m^2}\right)t}\partial_r\partial_{r_1}\left\langle r\sqrt{r^2+m^2}f_{+}\left(r\omega\right),\right. \\ & \times\left.\tilde{\zeta}_R\left(r\omega-r_1\omega'\right)r_1\sqrt{r_1^2+m^2}g_{+}\left(r_1\omega'\right)\right\rangle drdr_1d\omega'd\omega. \end{aligned}$$

Noting that

$$\left|\nabla^j\tilde{\zeta}_R\left(p\right)\right|\leq C_j,\ p\in\mathbb{R}^3,\ \text{for all } j,$$

we obtain

$$\left|\int_1^\infty h\left(f_{+},g_{+};t\right)dt\right|\leq C\left(\left\|\hat{f}\right\|_{L^\infty(|p|\leq1)}+\left\|\hat{f}\right\|_{\mathcal{H}^1_{3/2+\varepsilon}(\mathbb{R}^3)}\right)\left(\left\|\hat{g}\right\|_{L^\infty(|p|\leq1)}+\left\|\hat{g}\right\|_{\mathcal{H}^1_{3/2+\varepsilon}(\mathbb{R}^3)}\right)$$

and then, using the Sobolev embedding theorem, we get (3.3) for $f,g\in\mathcal{S}$. Hence, by continuity, we extend the estimate (3.3) for all $f,g\in\mathcal{H}^{3/2+\varepsilon}_{3/2+\varepsilon}$ and therefore, we arrive to (3.2). ∎

We now study the asymptotics of $I\left(R\right)$, as $R\to\infty$. We have the following

Theorem 3.2 *Let $f,g\in\mathcal{H}_2^{3/2+\varepsilon}$, $\varepsilon>0$. Then, as $R\to\infty$,*

$$\begin{aligned} I\left(R\right) &= R\int_{\mathbb{R}^3}\left\langle\frac{\sqrt{|p|^2+m^2}}{|p|}f_{+}\left(p\right),g_{+}\left(p\right)\right\rangle dp+R\int_{\mathbb{R}^3}\left\langle\frac{\sqrt{|p|^2+m^2}}{|p|}f_{-}\left(p\right),g_{-}\left(p\right)\right\rangle dp \\ &+i\int_{\mathbb{R}^3}\left\langle f_{+}\left(p\right),\frac{\sqrt{|p|^2+m^2}}{2|p|^2}g_{+}\left(p\right)+\frac{\sqrt{|p|^2+m^2}}{2|p|^2}p\cdot\nabla\left(g_{+}\left(p\right)\right)+\frac{1}{2|p|^2}p\cdot\nabla\left(\sqrt{|p|^2+m^2}g_{+}\left(p\right)\right)\right\rangle dp \\ &-i\int_{\mathbb{R}^3}\left\langle f_{-}\left(p\right),\frac{\sqrt{|p|^2+m^2}}{2|p|^2}g_{-}\left(p\right)+\frac{\sqrt{|p|^2+m^2}}{2|p|^2}p\cdot\nabla\left(g_{-}\left(p\right)\right)+\frac{1}{2|p|^2}p\cdot\nabla\left(\sqrt{|p|^2+m^2}g_{-}\left(p\right)\right)\right\rangle dp+o\left(1\right), \end{aligned} \tag{3.5}$$

where $f_{\pm}\left(p\right)=P_{\pm}\left(p\right)\hat{f}\left(p\right)$ and $g_{\pm}\left(p\right)=P_{\pm}\left(p\right)\hat{g}\left(p\right)$.

Proof. We decompose $I\left(R\right)$ as in relation (3.1). Let us consider the term I_1. Let $\varphi\left(s\right)\in C_0^\infty\left(\mathbb{R}\right)$, be such that $\varphi\left(s\right)=1$ in some neighborhood of $s=0$. Observe that

$$\begin{aligned} I_1\left(R\right) = \lim_{\varepsilon\to0}\int_0^\infty\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} & e^{-i(|p|-|q|)t}\varphi\left(\varepsilon\frac{\sqrt{|p|^2+m^2}+\sqrt{|q|^2+m^2}}{|p|+|q|}t\right) \\ & \times\left\langle\frac{\sqrt{|p|^2+m^2}+\sqrt{|q|^2+m^2}}{|p|+|q|}f_{+}\left(p\right),\tilde{\zeta}_R\left(p-q\right)g_{+}\left(q\right)\right\rangle dqdpdt. \end{aligned}$$

Proceeding as in (3.4), we show that

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}e^{-i(|p|-|q|)t}\varphi\left(\varepsilon\frac{\sqrt{|p|^2+m^2}+\sqrt{|q|^2+m^2}}{|p|+|q|}t\right)\left\langle\frac{\sqrt{|p|^2+m^2}+\sqrt{|q|^2+m^2}}{|p|+|q|}f_{+}\left(p\right),\tilde{\zeta}_R\left(p-q\right)g_{+}\left(q\right)\right\rangle dqdp$$

belongs to L^1, as a function of t, uniformly on $\varepsilon \leq 1$. Then, it follows from the dominated convergence theorem that

$$I_1(R) = \int_0^\infty \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left\langle e^{-i(|p|-|q|)t} \frac{\sqrt{|p|^2+m^2}+\sqrt{|q|^2+m^2}}{|p|+|q|} f_+(p), \tilde{\zeta}_R(p-q) g_+(q) \right\rangle dq dp dt.$$

Let the function $F(s)$, $s \in \mathbb{R}$, be such that $F(s) = 1$ for $0 \leq s < \infty$ and $F(s) = 0$ for $s < 0$. Passing to the spherical coordinate system in the q variable in the expression for $I_1(R)$, we get

$$I_1(R) = \lim_{\tau \to +0} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \int_0^\infty \left(\int_0^\infty e^{-i(|p|-r)t} e^{-\tau t} dt \right) \left\langle \frac{\sqrt{|p|^2+m^2}+\sqrt{r^2+m^2}}{|p|+r} f_+(p), \tilde{\zeta}_R(p-r\omega) g_+(r\omega) \right\rangle r^2 dr dp d\omega$$

$$= \lim_{\tau \to +0} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \int_{-\infty}^\infty \left(\int_0^\infty e^{-irt} e^{-\tau t} dt \right) f((|p|-r)\omega, p) \, dr dp d\omega,$$

where

$$f(q,p) := \left\langle \frac{\sqrt{|p|^2+m^2}+\sqrt{|q|^2+m^2}}{|p|+|q|} f_+(p), \tilde{\zeta}_R(p-q) g_+(q) \right\rangle F(|q|) |q|^2.$$

Moreover, as

$$\lim_{\tau \to +0} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \int_{-\infty}^\infty \left(\int_0^\infty e^{-irt} e^{-\tau t} dt \right) f((|p|-r)\omega, p) \, dr d\omega dp = -\lim_{\tau \to +0} i \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \int_{-\infty}^\infty \frac{f((|p|-r)\omega, p)}{r - i\tau} dr d\omega dp$$

$$= -\lim_{\delta \to 0} \lim_{\tau \to +0} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} i f(|p|\omega, p) \left(\int_{-\delta}^\delta \frac{dr}{r - i\tau} \right) d\omega dp - i \lim_{\delta \to 0} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \left[\int_{-\infty}^{-\delta} + \int_\delta^\infty \right] \frac{f((|p|-r)\omega, p)}{r} dr d\omega dp$$

$$- \lim_{\delta \to 0} \lim_{\tau \to +0} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} i \int_{-\delta}^\delta \frac{f((|p|-r)\omega, p) - f(|p|\omega, p)}{r - i\tau} dr d\omega dp,$$

we conclude that

$$I_1(R) = I_{1,1}(R) + I_{1,2}(R),$$

with

$$I_{1,1}(R) := \pi \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} f(|p|\omega, p) \, d\omega dp$$

and

$$I_{1,2}(R) := -i \lim_{\delta \to 0} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \left[\int_{-\infty}^{-\delta} + \int_\delta^\infty \right] \frac{f((|p|-r)\omega, p)}{r} dr d\omega dp.$$

Using Lemma 3.3 for $I_{1,1}(R)$ we obtain the first term in the R.H.S. of the asymptotic expansion (3.5). Decomposing $I_{1,2}(R)$ as in the sum (3.15) and applying Lemma 3.4 to $I_{1,2}^1(R)$, Lemma 3.5 to $I_{1,2}^2(R)$ and Lemma 3.6 to $I_{1,2}^3(R)$ we get the third term in the R.H.S. of (3.5).

Now note that

$$I_4(R) = \lim_{\tau \to +0} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \int_0^\infty \left(\int_0^\infty e^{i(|p|-r)t} e^{-\tau t} dt \right) \left\langle \frac{\sqrt{|p|^2+m^2}+\sqrt{r^2+m^2}}{|p|+r} f_-(p), \tilde{\zeta}_R(p-r\omega) g_-(r\omega) \right\rangle r^2 dr dp d\omega$$

$$= \lim_{\tau \to +0} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \int_{-\infty}^\infty \left(\int_0^\infty e^{irt} e^{-\tau t} dt \right) f_1((|p|-r)\omega, p) \, dr dp d\omega = \lim_{\tau \to +0} i \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \int_{-\infty}^\infty \frac{f_1((|p|-r)\omega, p)}{r + i\tau} dr d\omega dp$$

$$= i \lim_{\delta \to 0} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} \left[\int_{-\infty}^{-\delta} + \int_\delta^\infty \right] \frac{f_1((|p|-r)\omega, p)}{r} dr d\omega dp + \lim_{\delta \to 0} \lim_{\tau \to +0} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} i \int_{-\delta}^\delta \frac{f_1((|p|-r)\omega, p) - f_1(|p|\omega, p)}{r + i\tau} dr d\omega dp$$

$$+ \lim_{\delta \to 0} \lim_{\tau \to +0} \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} i f_1(|p|\omega, p) \left(\int_{-\delta}^\delta \frac{dr}{r + i\tau} \right) d\omega dp,$$

where $f_1(q,p) := \left\langle \frac{\sqrt{|p|^2+m^2}+\sqrt{|q|^2+m^2}}{|p|+|q|} f_-(p), \tilde{\zeta}_R(p-q)\, g_-(q) \right\rangle F(|q|)\,|q|^2$, and then

$$I_4(R) = I_{4,1}(R) + I_{4,2}(R),$$

where

$$I_4(R) := \pi \int\limits_{\mathbb{R}^3} \int\limits_{\mathbb{S}^2} f_1(|p|\,\omega, p)\, d\omega dp$$

and

$$I_{4,2}(R) := i \lim_{\delta\to 0} \int\limits_{\mathbb{R}^3} \int\limits_{\mathbb{S}^2} \left[\int\limits_{-\infty}^{-\delta} + \int\limits_{\delta}^{\infty} \right] \frac{f_1((|p|-r)\,\omega, p)}{r} dr d\omega dp.$$

Thus, similarly to the case of $I_1(R)$ above, we obtain the second and fourth terms in the R.H.S. of the asymptotic expansion (3.5). Finally, applying Lemma 3.7 to $I_2(R)$ and $I_3(R)$, we complete the proof. ■

Let us now check the results that we use in the proof of Theorem 3.2. First, we calculate the asymptotics of $I_{1,1}(R)$ as $R\to\infty$. We have

Lemma 3.3 *Suppose that $f, g \in \mathcal{H}_2^{3/2+\varepsilon}$, $\varepsilon > 0$. The following relation holds*

$$I_{1,1}(R) = 8\pi^3 R \int\limits_{\mathbb{R}^3} \left\langle \frac{\sqrt{|p|^2+m^2}}{|p|} f_+(p), g_+(p) \right\rangle dp + o(1), \quad \text{as } R\to\infty. \tag{3.6}$$

Proof. Noting that $\tilde{\zeta}_R(p) = 4\pi\left(-R\frac{\cos R|p|}{|p|^2} + \frac{\sin R|p|}{|p|^3}\right)$ (see Theorem 56, page 235 of [11]) and passing to the spherical coordinate system in ω, where the $z-$axis is directed along the vector p, we have

$$I_{1,1}(R) = 4\pi^2 \int\limits_{\mathbb{R}^3} \int\limits_0^{2\pi} \int\limits_0^{\pi} \left\langle f_0(p), \left(-R\frac{\cos\left(R|p|\sqrt{2-2\cos\theta}\right)}{(2-2\cos\theta)} + \frac{\sin\left(R|p|\sqrt{2-2\cos\theta}\right)}{|p|(2-2\cos\theta)^{\frac{3}{2}}} \right) g_+(|p|\,\omega(\theta,\varphi)) \right\rangle \sin\theta d\theta d\varphi dp,$$

where $f_0(p) := \frac{\sqrt{|p|^2+m^2}}{|p|} f_+(p)$ and $\omega(\theta,\varphi) := (\cos\varphi\sin\theta, \sin\varphi\sin\theta, \cos\theta)$. Observing that

$$\left(-R\frac{\cos\left(R|p|\sqrt{2-2\cos\theta}\right)}{(2-2\cos\theta)} + \frac{\sin\left(R|p|\sqrt{2-2\cos\theta}\right)}{|p|(2-2\cos\theta)^{\frac{3}{2}}} \right) \sin\theta = -\partial_\theta \frac{\sin\left(R|p|\sqrt{2-2\cos\theta}\right)}{|p|\sqrt{2-2\cos\theta}}$$

and integrating by parts in θ we get

$$I_{1,1}(R) = 8\pi^3 R \int\limits_{\mathbb{R}^3} \langle f_0(p), g_+(p)\rangle\, dp + I_{1,1}^1(R) + I_{1,1}^2(R), \tag{3.7}$$

where

$$I_{1,1}^1(R) := -8\pi^3 \int\limits_{\mathbb{R}^3} \sin(2R|p|) \left\langle f_0(p), \frac{g_+(-p)}{2|p|} \right\rangle dp, \tag{3.8}$$

and

$$I_{1,1}^2(R) := 4\pi^2 \int\limits_{\mathbb{R}^3} \int\limits_0^{2\pi} \int\limits_0^{\pi} \left\langle f_0(p), \frac{\sin\left(R|p|\sqrt{2-2\cos\theta}\right)}{|p|\sqrt{2-2\cos\theta}} \partial_\theta g_+(|p|\,\omega(\theta,\varphi)) \right\rangle d\theta d\varphi dp.$$

Note that

$$\int\limits_{\mathbb{R}^3} |\langle f_0(p), g_+(-p)\rangle| \frac{1}{2|p|} dp \le C\left(\|f_+\|_{L^\infty(|p|\le 1)} + \|f_+\|_{L^2(\mathbb{R}^3)} \right) \left(\|g_+\|_{L^\infty(|p|\le 1)} + \|g_+\|_{L^2(\mathbb{R}^3)} \right) < \infty. \tag{3.9}$$

Taking $p = |p|\,\omega$ in the integral in (3.8), it follows from (3.9) and Fubini's theorem that

$$\int\limits_0^{\infty} \left\langle f_0(p), \frac{g_+(-p)}{2|p|} \right\rangle |p|^2\, d|p| \in L^1\left(\mathbb{S}^2\right).$$

In particular, we have

$$\int_0^\infty \left\langle f_0(p), \frac{g_+(-p)}{2|p|} \right\rangle |p|^2 \, d|p| < \infty,$$

for almost all $\omega \in \mathbb{S}^2$. Then by the Riemann-Lebesgue lemma we get

$$\lim_{R\to\infty} \int_0^\infty \sin(2R|p|) \left\langle f_0(p), \frac{g_+(-p)}{2|p|} \right\rangle |p|^2 \, d|p| = 0,$$

a.e. in $\omega \in \mathbb{S}^2$. Thus, using (3.9) to apply the dominated convergence theorem in (3.8) we obtain

$$\lim_{R\to\infty} I_{1,1}^1(R) = 0. \tag{3.10}$$

Let us consider now $I_{1,1}^2(R)$. Note that

$$\begin{aligned} I_{1,1}^2(R) = {}& 4\pi^2 \int_{\mathbb{R}^3} \int_0^{2\pi} \int_0^{\pi} \left\langle f_0(p), \frac{\sin\left(R|p|\sqrt{2-2\cos\theta}\right)}{2\sqrt{2-2\cos\theta}} (\cos\varphi, \sin\varphi, 0) \cdot \nabla g_+(|p|\,\omega(\theta,\varphi)) \right\rangle \cos\theta d\theta d\varphi dp \\ & -4\pi^2 \int_{\mathbb{R}^3} \int_0^{2\pi} \int_0^{\pi} \left\langle f_0(p), \frac{\sqrt{2+2\cos\theta}\sin\left(R|p|\sqrt{2-2\cos\theta}\right)}{2} (0,0,1) \cdot \nabla g_+(|p|\,\omega(\theta,\varphi)) \right\rangle d\theta d\varphi dp. \end{aligned} \tag{3.11}$$

We have

$$\int_{\mathbb{R}^3} |f_0(p)| \int_0^{2\pi} \int_0^{\pi} |\nabla g_+(|p|\,\omega(\theta,\varphi))| \, d\theta d\varphi dp \le C \left(\|f_+\|_{L^\infty(|p|\le 1)} + \|f_+\|_{L^2(\mathbb{R}^3)} \right) \|g_+\|_{\mathcal{H}^1(\mathbb{R}^3)}.$$

Then, arguing as in the case of (3.10), we get

$$\lim_{R\to\infty} 4\pi^2 \int_{\mathbb{R}^3} \int_0^{2\pi} \int_0^{\pi} \left\langle f_0(p), \frac{\sqrt{2+2\cos\theta}\sin\left(R|p|\sqrt{2-2\cos\theta}\right)}{2} (0,0,1) \cdot \nabla g_+(|p|\,\omega(\theta,\varphi)) \right\rangle d\theta d\varphi dp = 0. \tag{3.12}$$

Observe now that

$$\begin{aligned} & 4\pi^2 \int_{\mathbb{R}^3} \int_0^{2\pi} \int_0^{\pi} \left\langle f_0(p), \frac{\sin\left(R|p|\sqrt{2-2\cos\theta}\right)}{2\sqrt{2-2\cos\theta}} (\cos\varphi, \sin\varphi, 0) \cdot \nabla g_+(|p|\,\omega(\theta,\varphi)) \right\rangle \cos\theta d\theta d\varphi dp \\ & = 4\pi^2 \int_{\mathbb{R}^3} \int_0^{2\pi} \int_0^{\pi} \left\langle f_0(p), \frac{\sin\left(R|p|\sqrt{2-2\cos\theta}\right)}{2\sqrt{2-2\cos\theta}} (\cos\varphi, \sin\varphi, 0) \cdot \left(\nabla g_+(|p|\,\omega(\theta,\varphi)) - \nabla g_+(p)\right) \right\rangle \cos\theta d\theta d\varphi dp. \end{aligned}$$

Thus, as

$$\begin{aligned} & \int_{\mathbb{R}^3} \int_0^{2\pi} \int_0^{\pi} \frac{|f_0(p)|}{\sqrt{2-2\cos\theta}} |\nabla g_+(|p|\,\omega(\theta,\varphi)) - \nabla g_+(p)| \, d\theta d\varphi dp \\ & \le C \left(\int_0^{\pi} \frac{|\theta|^{1/2}}{\sqrt{2-2\cos\theta}} d\theta \right) \int_{\mathbb{R}^3} |f_0(p)| \left(\int_0^{2\pi} \int_0^{\pi} |\partial_\theta (\nabla g_+(|p|\,\omega(\theta,\varphi)))|^2 \, d\theta d\varphi \right)^{1/2} dp \le C \|f_+\|_{L_1^2(\mathbb{R}^3)} \|g_+\|_{\mathcal{H}^2(\mathbb{R}^3)}, \end{aligned}$$

arguing as in (3.10), we see that the limit of the first term in the R.H.S. of (3.11), as $R \to \infty$, is equals to 0. Therefore, passing to the limit, as $R \to \infty$, in (3.11) and using (3.12) we conclude that

$$\lim_{R\to\infty} I_{1,1}^2(R) = 0. \tag{3.13}$$

Using relations (3.10) and (3.13) in (3.7) we obtain (3.6). ■

Next we study the asymptotics of $I_{1,2}(R)$ as $R\to\infty$. Passing to the spherical coordinate system, where the $z-$axis is directed along the vector p, we obtain

$$I_{1,2}(R)=-4\pi i\lim_{\delta\to0}\int_{\mathbb{R}^3}\int_0^{2\pi}\int_0^{\pi}\left[\int_{-\infty}^{|p|-\delta}+\int_{|p|+\delta}^{\infty}\right]$$
$$\times\frac{\left\langle f_+(p),\left(-R\frac{\cos\left(R\sqrt{|p|^2-2r|p|\cos\theta+r^2}\right)}{|p|^2-2r|p|\cos\theta+r^2}+\frac{\sin\left(R\sqrt{|p|^2-2r|p|\cos\theta+r^2}\right)}{\left(|p|^2-2r|p|\cos\theta+r^2\right)^{3/2}}\right)g_0(r\omega(\theta,\varphi);|p|)\sin\theta\right\rangle}{(|p|-r)(|p|+r)}F(r)\,r^2drd\theta d\varphi dp,$$

where $g_0(q;|p|):=\left(\sqrt{|p|^2+m^2}+\sqrt{|q|^2+m^2}\right)g_+(q)$ and $\omega(\theta,\varphi)=(\cos\varphi\sin\theta,\sin\varphi\sin\theta,\cos\theta)$. Noting that

$$\left(-R\frac{\cos\left(R\sqrt{|p|^2-2r|p|\cos\theta+r^2}\right)}{|p|^2-2r|p|\cos\theta+r^2}+\frac{\sin\left(R\sqrt{|p|^2-2r|p|\cos\theta+r^2}\right)}{\left(|p|^2-2r|p|\cos\theta+r^2\right)^{3/2}}\right)\sin\theta=-\partial_\theta\frac{\sin\left(R\sqrt{|p|^2-2r|p|\cos\theta+r^2}\right)}{r|p|\sqrt{|p|^2-2r|p|\cos\theta+r^2}}$$

we have

$$\int_0^{\pi}\tilde{\zeta}_R(p-r\omega(\theta,\varphi))\,g_0(r\omega(\theta,\varphi);|p|)\sin\theta d\theta$$
$$=\int_0^{\pi}\left(-R\frac{\cos\left(R\sqrt{|p|^2-2r|p|\cos\theta+r^2}\right)}{|p|^2-2r|p|\cos\theta+r^2}+\frac{\sin\left(R\sqrt{|p|^2-2r|p|\cos\theta+r^2}\right)}{\left(|p|^2-2r|p|\cos\theta+r^2\right)^{3/2}}\right)g_0(r\omega(\theta,\varphi);|p|)\sin\theta d\theta$$
$$=-\int_0^{\pi}\partial_\theta\frac{\sin\left(R\sqrt{|p|^2-2r|p|\cos\theta+r^2}\right)}{r|p|\sqrt{|p|^2-2r|p|\cos\theta+r^2}}g_0(r\omega(\theta,\varphi);|p|)\,d\theta$$
$$=-\frac{\sin(R(|p|+r))}{r|p|(|p|+r)}g_0\left(-r\tfrac{p}{|p|};|p|\right)+\frac{\sin(R(|p|-r))}{r|p|(|p|-r)}g_0\left(r\tfrac{p}{|p|};|p|\right)+\int_0^{\pi}\frac{\sin\left(R\sqrt{|p|^2-2r|p|\cos\theta+r^2}\right)}{r|p|\sqrt{|p|^2-2r|p|\cos\theta+r^2}}\partial_\theta g_0(r\omega(\theta,\varphi);|p|)\,d\theta,\tag{3.14}$$

and therefore, we obtain the following decomposition

$$I_{1,2}(R)=I_{1,2}^1(R)+I_{1,2}^2(R)+I_{1,2}^3(R),\tag{3.15}$$

with

$$I_{1,2}^1(R):=8\pi^2 i\lim_{\delta\to0}\int_{\mathbb{R}^3}\left[\int_{-\infty}^{|p|-\delta}+\int_{|p|+\delta}^{\infty}\right]\frac{\sin(R(|p|+r))}{|p|(|p|-r)(|p|+r)^2}\left\langle f_+(p),g_0\left(-r\frac{p}{|p|};|p|\right)\right\rangle F(r)\,rdrdp,$$

$$I_{1,2}^2(R):=-8\pi^2 i\lim_{\delta\to0}\int_{\mathbb{R}^3}\left[\int_{-\infty}^{|p|-\delta}+\int_{|p|+\delta}^{\infty}\right]\frac{\sin(R(|p|-r))}{|p|(|p|-r)^2}\left\langle f_+(p),\frac{g_0\left(r\frac{p}{|p|};|p|\right)}{|p|+r}\right\rangle F(r)\,rdrdp$$

and

$$I_{1,2}^3(R):=-4\pi i\lim_{\delta\to0}\int_{\mathbb{R}^3}\left[\int_{-\infty}^{|p|-\delta}+\int_{|p|+\delta}^{\infty}\right]\frac{\left(\sqrt{|p|^2+m^2}+\sqrt{r^2+m^2}\right)}{r|p|(|p|-r)(|p|+r)}$$
$$\times\left\langle f_+(p),\int_0^{2\pi}\int_0^{\pi}\frac{\sin\left(R\sqrt{|p|^2-2r|p|\cos\theta+r^2}\right)}{\sqrt{|p|^2-2r|p|\cos\theta+r^2}}\partial_\theta g_+(r\omega(\theta,\varphi))\,d\theta d\varphi\right\rangle F(r)\,r^2drdp.$$

For $I_{1,2}^1(R)$ we have

Lemma 3.4 *Let $f,g\in\mathcal{H}_{3/2+\varepsilon}^{3/2+\varepsilon}$, $\varepsilon>0$. Then,*

$$\lim_{R\to\infty}I_{1,2}^1(R)=0.\tag{3.16}$$

Proof. Note that

$$I_{1,2}^{1}(R)=8\pi^{2}i\lim_{\delta\to0}\int_{\mathbb{R}^{3}}\left\langle f_{+}(p),g_{0}(-p;|p|)\right\rangle i_{1,2}^{1}(R,|p|,\delta)\,dp$$
$$+8\pi^{2}i\lim_{\delta\to0}\int_{\mathbb{R}^{3}}\left[\int_{-\infty}^{|p|-\delta}+\int_{|p|+\delta}^{\infty}\right]\frac{\sin(R(|p|+r))}{|p|(|p|+r)^{2}(|p|-r)}\left\langle f_{+}(p),\int_{|p|}^{r}\partial_{r_{1}}\left(g_{0}\left(-r_{1}\tfrac{p}{|p|};|p|\right)r_{1}\right)dr_{1}\right\rangle F(r)\,drdp, \tag{3.17}$$

where

$$i_{1,2}^{1}(R,|p|,\delta):=\left[\int_{-\infty}^{|p|-\delta}+\int_{|p|+\delta}^{\infty}\right]\frac{\sin(R(|p|+r))}{(|p|-r)(|p|+r)^{2}}F(r)\,dr.$$

Since for any $\varepsilon>0$,

$$\int_{\mathbb{R}^{3}}\int_{0}^{\infty}\frac{1}{(|p|+r)^{2}\left||r-|p|\right|}\left|\left\langle\frac{f_{+}(p)}{|p|},\int_{|p|}^{r}\partial_{r_{1}}\left(g_{0}\left(-r_{1}\tfrac{p}{|p|};|p|\right)r_{1}\right)dr_{1}\right\rangle\right|drdp$$
$$\leq\int_{0}^{\infty}\int_{0}^{\infty}\frac{dr}{(|p|+r)^{2}\left|r-|p|\right|^{1/2}}\int_{\mathbb{S}^{2}}\left|\frac{f_{+}(|p|\omega)}{|p|}\right|\left(\int_{0}^{\infty}\left|\partial_{r_{1}}\left(g_{0}(-r_{1}\omega;|p|)r_{1}\right)\right|^{2}dr_{1}\right)^{1/2}d\omega\,|p|^{2}\,d|p|$$
$$\leq\left(\int_{0}^{\infty}\int_{\mathbb{S}^{2}}\left|\partial_{r_{1}}\left(g_{0}(r_{1}\omega;|p|)r_{1}\right)\right|^{2}d\omega dr_{1}\right)^{1/2}\int_{0}^{\infty}\int_{0}^{\infty}\frac{dr}{(|p|+r)^{2}\left|r-|p|\right|^{1/2}}\left(\int_{\mathbb{S}^{2}}\left|\frac{f_{+}(|p|\omega)}{|p|}\right|^{2}d\omega\right)^{1/2}|p|^{2}\,d|p|$$
$$\leq C\left(\|f_{+}\|_{L^{\infty}(|p|\leq1)}+\|f_{+}\|_{L^{2}_{\varepsilon}(\mathbb{R}^{3})}\right)\left(\|g_{+}\|_{L^{\infty}(|p|\leq1)}+\|g_{+}\|_{\mathcal{H}^{1}}\right),$$

arguing as in (3.10) we get

$$8\pi^{2}i\lim_{R\to\infty}\lim_{\delta\to0}\int_{\mathbb{R}^{3}}\left[\int_{-\infty}^{|p|-\delta}+\int_{|p|+\delta}^{\infty}\right]\frac{\sin(R(|p|+r))}{|p|(|p|+r)^{2}(|p|-r)}\left\langle f_{+}(p),\int_{|p|}^{r}\partial_{r_{1}}g_{0}\left(-r_{1}\frac{p}{|p|};|p|\right)dr_{1}\right\rangle F(r)\,rdrdp=0. \tag{3.18}$$

For $\delta<1$, we decompose $i_{1,2}^{1}(R,|p|,\delta)$ in the sum

$$i_{1,2}^{1}(R,|p|,\delta)=-\frac{\cos2R|p|}{(2|p|)^{2}}\left[\int_{-1}^{-\delta}+\int_{\delta}^{1}\right]\frac{\sin(Rr)}{r}F(|p|-r)\,dr+\sin(2R|p|)\left(i_{1,2}^{1}(R,|p|,\delta)\right)_{1}$$
$$+\left[\int_{-1}^{-\delta}+\int_{\delta}^{1}\right]\sin(R(2|p|-r))\frac{1}{r}\left[\frac{1}{(2|p|-r)^{2}}-\frac{1}{(2|p|)^{2}}\right]F(|p|-r)\,dr+\left[\int_{-\infty}^{-1}+\int_{1}^{\infty}\right]\frac{\sin R(2|p|-r)}{r(2|p|-r)^{2}}F(|p|-r)\,dr,$$

where

$$\left(i_{1,2}^{1}(R,|p|,\delta)\right)_{1}:=\frac{1}{(2|p|)^{2}}\left[\int_{-1}^{-\delta}+\int_{\delta}^{1}\right]\frac{\cos(Rr)}{r}F(|p|-r)\,dr.$$

For $|p|<\delta<1$ and any $\varepsilon>0$ we get

$$\left|\left(i_{1,2}^{1}(R,|p|,\delta)\right)_{1}\right|\leq\frac{\ln\delta}{(2|p|)^{2}}\leq\frac{\delta^{\varepsilon}\ln\delta}{(2|p|)^{2+\varepsilon}}.$$

If $\delta\leq|p|<1$,

$$\left(i_{1,2}^{1}(R,|p|,\delta)\right)_{1}=\frac{\sin(2R|p|)}{(2|p|)^{2}}\int_{-1}^{-|p|}\frac{\cos Rr}{r}dr,$$

and thus,

$$\left|\left(i_{1,2}^{1}(R,|p|,\delta)\right)_{1}\right|\leq\frac{\left|\ln|p|\right|}{(2|p|)^{2}}.$$

Finally, for $|p| \geq 1$,

$$\left(i^1_{1,2}(R, |p|, \delta)\right)_1 = 0.$$

From these relations, together with the estimates

$$\frac{1}{(2|p|)^2}\left|\left[\int_{-1}^{-\delta} + \int_{\delta}^{1}\right]\frac{\sin(Rr)}{r}F(|p|-r)\,dr\right| = \frac{1}{(2|p|)^2}\left|\left[\int_{-R}^{-R\delta} + \int_{R\delta}^{R}\right]\frac{\sin r}{r}F\left(|p|-\frac{r}{R}\right)dr\right| \leq C\frac{1}{|p|^2},$$

uniformly for R and δ (since $\int_{-\infty}^{\infty}\frac{\sin r}{r}dr = 2\int_0^{\infty}\frac{\sin r}{r}dr = \pi$ implies that $\int_a^b \frac{\sin r}{r}dr \leq C$, for all a and b),

$$\int_{-1}^{1}\left|\frac{1}{r}\left[\frac{1}{(2|p|-r)^2} - \frac{1}{(2|p|)^2}\right]\right| F(|p|-r)\,dr \leq \int_{-1}^{|p|}\left|\frac{1}{r}\left[\frac{4r|p|-r^2}{(2|p|-r)^2(2|p|)^2}\right]\right| dr$$
$$\leq \int_{-1}^{|p|}\left[\frac{1}{(2|p|-r)^2(2|p|)}\right] + \left[\frac{1}{(2|p|-r)(2|p|)^2}\right]dr \leq C\frac{(1+\ln|p|)}{|p|^2}$$

and

$$\left[\int_{-\infty}^{-1} + \int_{1}^{\infty}\right]\frac{1}{|r|(2|p|-r)^2}F(|p|-r)\,dr \leq C\left(1+\frac{1}{|p|^2}\right),$$

we obtain

$$\int_{\mathbb{R}^3}\left|\langle f_+(p), g_0(-p;|p|)\rangle\, i^1_{1,2}(R,|p|,\delta)\right| dp \leq C\|f_+\|_{L^2}\|g_+\|_{L^2_1}$$
$$+C\left(\|f_+\|_{L^\infty(|p|\leq 1)} + \|f_+\|_{L^2(\mathbb{R}^3)}\right)\left(\|g_+\|_{L^\infty(|p|\leq 1)} + \|g_+\|_{L^2(\mathbb{R}^3)}\right).$$

Arguing as in (3.10) we get

$$8\pi^2 i \lim_{R\to\infty}\lim_{\delta\to 0}\int_{\mathbb{R}^3}\langle f_+(p), g_0(-p;|p|)\rangle\, i^1_{1,2}(R,|p|,\delta)\,dp = 0. \tag{3.19}$$

Moreover, taking the limit in (3.17), as $R\to\infty$, and using (3.19) and (3.18) we obtain (3.16). ■

The following result shows that the term $I^2_{1,2}(R)$ gives the non zero part of the asymptotics of $I_{1,2}(R)$ as $R\to\infty$

Lemma 3.5 *For $f, g \in \mathcal{H}_2^{3/2+\varepsilon}$, $\varepsilon > 0$ we have*

$$\lim_{R\to\infty} I^2_{1,2}(R) = 8\pi^3 i\int_{\mathbb{R}^3}\left\langle f_+(p), \frac{g_0(p;|p|)}{4|p|^2} + \frac{p\cdot(\nabla_q g_0)(p;|p|)}{2|p|^2}\right\rangle dp, \tag{3.20}$$

where $g_0(q;|p|) = \left(\sqrt{|p|^2+m^2} + \sqrt{|q|^2+m^2}\right)g_+(q)$.

Proof. Observe that

$$I^2_{1,2}(R) = I^{2,1}_{1,2}(R) + I^{2,2}_{1,2}(R), \tag{3.21}$$

where

$$I^{2,1}_{1,2}(R) := -8\pi^2 i\int_{\mathbb{R}^3}\left[\int_{-\infty}^{-1} + \int_{1}^{\infty}\right]\frac{\sin(Rr)}{|p|r^2}\left\langle f_+(p), \frac{g_0\left((|p|-r)\frac{p}{|p|};|p|\right)}{2|p|-r}\right\rangle F(|p|-r)(|p|-r)\,drdp,$$

and

$$I^{2,2}_{1,2}(R) := -8\pi^2 i\lim_{\delta\to 0}\int_{\mathbb{R}^3}\left[\int_{-1}^{-\delta} + \int_{\delta}^{1}\right]\frac{\sin(Rr)}{|p|r^2}\left\langle f_+(p), \frac{g_0\left((|p|-r)\frac{p}{|p|};|p|\right)}{2|p|-r}\right\rangle F(|p|-r)(|p|-r)\,drdp.$$

Since

$$\int_{\mathbb{R}^3}\left[\int_{-\infty}^{-1}+\int_{1}^{\infty}\right]\left|\frac{1}{|p|\,r^2}\left\langle f_+(p)\,,\frac{g_0\left((|p|-r)\frac{p}{|p|};|p|\right)}{2|p|-r}\right\rangle F(|p|-r)(|p|-r)\right|drdp$$
$$\leq C\int_0^\infty\int_0^\infty\int_{\mathbb{S}^2}\left|f_+(|p|\,\omega)\left(1+\frac{1}{|p|}\right)\right|\left|g_+(r\omega)\right|d\omega r dr\,|p|^2\,d|p|$$
$$\leq C\left(\int_0^\infty\left(\int_{\mathbb{S}^2}\left|\frac{f_+(|p|\,\omega)}{|p|}\right|^2d\omega\right)^{1/2}|p|^2\,d|p|\right)\left(\int_0^\infty\left(\int_{\mathbb{S}^2}|g_+(r\omega)|^2\,r^2d\omega\right)^{1/2}dr\right)$$
$$\leq C\left(\|f_+\|_{L^\infty(|p|\leq1)}+\|f_+\|_{L^2_{3/2+\varepsilon}}\right)\left(\|g_+\|_{L^\infty(|p|\leq1)}+\|g_+\|_{L^2_{3/2+\varepsilon}}\right),$$

arguing as in (3.10) we obtain

$$\lim_{R\to\infty}I_{1,2}^{2,1}(R)=0. \tag{3.22}$$

As for all $\delta\leq1$

$$\left[\int_{-1}^{-\delta}+\int_{\delta}^{1}\right]\int_{\mathbb{R}^3}\frac{1}{|p|^2\,r^2}\left|\left\langle f_+(p)\,,g_0\left((|p|-r)\tfrac{p}{|p|};|p|\right)\right\rangle F(|p|-r)\frac{|p|-r}{2|p|-r}\right|dpdr$$
$$\leq\frac{C}{\delta^2}\left(\|f_+\|_{L^\infty(|p|\leq1)}+\|f_+\|_{L^2_{1/2+\varepsilon}}\right)\|g_+\|_{L^\infty},$$

it follows from the Fubini's theorem that

$$I_{1,2}^{2,2}(R)=-8\pi^2 i\lim_{\delta\to0}\left[\int_{-1}^{-\delta}+\int_{\delta}^{1}\right]\sin(Rr)\int_{\mathbb{R}^3}\frac{1}{|p|\,r^2}\left\langle f_+(p)\,,\frac{g_0\left((|p|-r)\frac{p}{|p|};|p|\right)}{2|p|-r}\right\rangle F(|p|-r)(|p|-r)\,dpdr$$
$$=-8\pi^2 i\lim_{\delta\to0}\int_{\delta}^{1}\frac{\sin(Rr)}{r^2}\int_{|p|\leq\frac{r}{3}}\left(\left\langle f_+(p)\,,g_0\left((|p|+r)\tfrac{p}{|p|};|p|\right)\right\rangle\frac{|p|+r}{2|p|+r}\frac{dp}{|p|}\right)dr \tag{3.23}$$
$$-8\pi^2 i\lim_{\delta\to0}\left[\int_{-1}^{-\delta}+\int_{\delta}^{1}\right]\frac{\sin(Rr)}{r^2}\int_{|p|\geq\frac{|r|}{3}}\left(\left\langle f_+(p)\,,\frac{g_0\left((|p|-r)\frac{p}{|p|};|p|\right)}{2|p|-r}\right\rangle F(|p|-r)(|p|-r)\frac{dp}{|p|}\right)dr.$$

Noting that

$$\int_{|p|\leq\frac{r}{3}}\left|\left(\left\langle f_+(p)\,,g_0\left((|p|+r)\frac{p}{|p|};|p|\right)\right\rangle\frac{|p|+r}{2|p|+r}\right)\right|\frac{dp}{|p|}\leq Cr^2\,\|f_+\|_{L^\infty(|p|\leq1)}\,\|g_+\|_{L^\infty(|p|\leq2)},$$

we get

$$\int_0^1\frac{1}{r^2}\left|\int_{|p|\leq\frac{r}{3}}\left(\left\langle f_+(p)\,,g_0\left(|p|+r,\frac{p}{|p|}\right)\right\rangle\frac{|p|+r}{2|p|+r}\frac{dp}{|p|}\right)\right|dr\leq C,$$

and thus, arguing as in (3.10), we see that

$$\lim_{R\to\infty}\lim_{\delta\to0}\int_{\delta}^{1}\frac{\sin(Rr)}{r^2}\int_{|p|\leq\frac{r}{3}}\left(\left\langle f_+(p)\,,g_0\left((|p|+r)\frac{p}{|p|};|p|\right)\right\rangle\frac{|p|+r}{2|p|+r}\frac{dp}{|p|}\right)dr=0. \tag{3.24}$$

Using that

$$\frac{1}{2|p|-r}=\frac{1}{2|p|}+\frac{r}{4|p|^2}+\int_0^r\frac{1}{(2|p|-t)^3}(r-t)\,dt,$$

and

$$g_0\left((|p|-r)\frac{p}{|p|};|p|\right)=g_0(p;|p|)-\left(\frac{p\cdot(\nabla_q g_0)(p;|p|)}{|p|}\right)r+\frac{1}{2}\int_0^r\partial_t^2 g_0\left((|p|-t)\frac{p}{|p|};|p|\right)(r-t)\,dt,$$

we get

$$-8\pi^2 i\lim_{\delta\to 0}\left[\int_{-1}^{-\delta}+\int_{\delta}^{1}\right]\frac{\sin(Rr)}{r^2}\int_{|p|\geq\frac{|r|}{3}}\left(\left\langle f_+(p),\frac{g_0\left((|p|-r)\frac{p}{|p|};|p|\right)}{2|p|-r}\right\rangle F(|p|-r)(|p|-r)\frac{dp}{|p|}\right)dr \tag{3.25}$$
$$=J_1^{2,2}(R)+J_2^{2,2}(R)+J_3^{2,2}(R),$$

where

$$J_1^{2,2}(R):=-8\pi^2 i\lim_{\delta\to 0}\left[\int_{-1}^{-\delta}+\int_{\delta}^{1}\right]\frac{\sin(Rr)}{r^2}\int_{|p|\geq\frac{|r|}{3}}F(|p|-r)\left\langle f_+(p),\frac{g_0(p;|p|)}{2|p|}\right\rangle dpdr,$$

$$J_2^{2,2}(R):=8\pi^2 i\lim_{\delta\to 0}\left[\int_{-1}^{-\delta}+\int_{\delta}^{1}\right]\frac{\sin(Rr)}{r}\int_{|p|\geq\frac{|r|}{3}}F(|p|-r)\left\langle f_+(p),\frac{g_0(p;|p|)}{4|p|^2}+\frac{p\cdot(\nabla_q g_0)(p;|p|)}{2|p|^2}\right\rangle dpdr,$$

and

$$J_3^{2,2}(R):=8\pi^2 i\int_{-1}^{1}\sin(Rr)\int_{|p|\geq\frac{|r|}{3}}\frac{1}{|p|}F(|p|-r)\left\langle f_+(p),\frac{g_0(p;|p|)}{4|p|^2}-\frac{p\cdot(\nabla_q g_0)(p;|p|)}{2|p|^2}\right\rangle dpdr$$
$$-8\pi^2 i\int_{-1}^{1}\sin(Rr)\int_{|p|\geq\frac{|r|}{3}}\frac{1}{|p|\,r^2}\left\langle f_+(p),w\left(|p|,\tfrac{p}{|p|},r\right)\right\rangle F(|p|-r)(|p|-r)\,dpdr, \tag{3.26}$$

with

$$w\left(|p|,\tfrac{p}{|p|},r\right):=\left(\int_0^r\frac{1}{(2|p|-t)^3}(r-t)\,dt\right)g_0(p;|p|)$$
$$-\left(\tfrac{r}{4|p|^2}+\int_0^r\frac{1}{(2|p|-t)^3}(r-t)\,dt\right)\frac{p\cdot(\nabla_q g_0)(p;|p|)}{|p|}r+\frac{1}{2(2|p|-r)}\int_0^r\partial_t^2 g_0\left((|p|-t)\tfrac{p}{|p|};|p|\right)(r-t)\,dt.$$

Note that

$$J_1^{2,2}(R)=-8\pi^2 i\lim_{\delta\to 0}\left[\int_{-1}^{-\delta}+\int_{\delta}^{1}\right]\frac{\sin(Rr)}{r^2}\int_{|p|\geq|r|}\left\langle f_+(p),\frac{g_0(p;|p|)}{2|p|}\right\rangle dpdr$$
$$-8\pi^2 i\lim_{\delta\to 0}\left[\int_{-1}^{-\delta}+\int_{\delta}^{1}\right]\frac{\sin(Rr)}{r^2}\int_{\frac{|r|}{3}\leq|p|\leq|r|}F(|p|-r)\left\langle f_+(p),\frac{g_0(p;|p|)}{2|p|}\right\rangle dpdr$$
$$=8\pi^2 i\int_0^1\sin(Rr)\int_{\frac{r}{3}\leq|p|\leq r}\frac{1}{r^2}\left\langle f_+(p),\frac{g_0(p;|p|)}{2|p|}\right\rangle dpdr.$$

As for any $0<\varepsilon<1$

$$\int_0^1\int_{\frac{r}{3}\leq|p|\leq r}\frac{1}{r^2}\left|\left\langle f_+(p),\frac{g_0(p;|p|)}{2|p|}\right\rangle\right|dpdr\leq\int_0^1\frac{dr}{r^{1-\varepsilon}}\int_{|p|\leq 1}\frac{1}{|p|^{2+\varepsilon}}\left|\langle f_+(p),g_0(p;|p|)\rangle\right|dp$$
$$\leq\frac{C}{\varepsilon}\|f_+\|_{L^\infty(|p|\leq 1)}\|g_+\|_{L^\infty(|p|\leq 1)}$$

arguing as in (3.10) we get

$$\lim_{R\to\infty}J_1^{2,2}(R)=0. \tag{3.27}$$

We split $J_2^{2,2}(R)$ as

$$J_2^{2,2}(R)=8\pi^2 i\int_{-1}^{1}\sin(Rr)\int_{\mathbb{R}^3}\frac{1}{r}F(|p|-r)\left\langle f_+(p),\frac{g_0(p;|p|)}{4|p|^2}+\frac{p\cdot(\nabla_q g_0)(p;|p|)}{2|p|^2}\right\rangle dpdr$$
$$-8\pi^2 i\int_0^1\sin(Rr)\int_{|p|\leq\frac{r}{3}}\frac{1}{r}\left\langle f_+(p),\frac{g_0(p;|p|)}{4|p|^2}+\frac{p\cdot(\nabla_q g_0)(p;|p|)}{2|p|^2}\right\rangle dpdr. \tag{3.28}$$

Noting that for any $0 < \varepsilon < \frac{1}{2}$

$$\begin{aligned}
&\int_0^1 \frac{1}{r} \int_{|p| \leq \frac{r}{3}} \left| \left\langle f_+ (p), \frac{g_0 (p; |p|)}{4 |p|^2} + \frac{p \cdot (\nabla_q g_0) (p; |p|)}{2 |p|^2} \right\rangle \right| dpdr \\
&\leq \int_0^1 \frac{dr}{r^{1-\varepsilon}} \int_{|p| \leq 1} \frac{|f_+ (p)|}{|p|^{1+\varepsilon}} \left(\left| \frac{g_0 (p; |p|)}{4 |p|} \right| + \left| \frac{p \cdot (\nabla_q g_0) (p; |p|)}{|p|} \right| \right) dp \\
&\leq \frac{C}{\varepsilon} \|f_+\|_{L^\infty(|p| \leq 1)} \left(\|g_+\|_{L^\infty(|p| \leq 1)} + \|g_+\|_{\mathcal{H}^1(|p| \leq 1)} \right),
\end{aligned}$$

and arguing as in (3.10) we obtain

$$8\pi^2 i \lim_{R \to \infty} \int_0^1 \sin (Rr) \int_{|p| \leq \frac{r}{3}} \frac{1}{r} \left\langle f_+ (p), \frac{g_0 (p; |p|)}{4 |p|^2} + \frac{p \cdot (\nabla_q g_0) (p; |p|)}{2 |p|^2} \right\rangle dpdr = 0. \tag{3.29}$$

Observe now that

$$\begin{aligned}
&8\pi^2 i \int_{-1}^1 \int_{\mathbb{R}^3} \frac{\sin (Rr)}{r} F (|p| - r) \left\langle f_+ (p), \frac{g_0 (p; |p|)}{4 |p|^2} + \frac{p \cdot (\nabla_q g_0) (p; |p|)}{2 |p|^2} \right\rangle dpdr \\
&= 8\pi^2 i \int_{\mathbb{R}^3} \left(\int_{-1}^1 \frac{\sin (Rr)}{r} F (|p| - r) \, dr \right) \left\langle f_+ (p), \frac{g_0 (p; |p|)}{4 |p|^2} + \frac{p \cdot (\nabla_q g_0) (p; |p|)}{2 |p|^2} \right\rangle dp \\
&= 8\pi^2 i \int_{|p| \geq 1} \left(\int_{-R}^R \frac{\sin r}{r} dr \right) \left\langle f_+ (p), \frac{g_0 (p; |p|)}{4 |p|^2} + \frac{p \cdot (\nabla_q g_0) (p; |p|)}{2 |p|^2} \right\rangle dp \\
&+ 8\pi^2 i \int_{|p| \leq 1} \left(\int_{-R}^{R|p|} \frac{\sin r}{r} dr \right) \left\langle f_+ (p), \frac{g_0 (p; |p|)}{4 |p|^2} + \frac{p \cdot (\nabla_q g_0) (p; |p|)}{2 |p|^2} \right\rangle dp.
\end{aligned}$$

Since

$$\begin{aligned}
&\int_{|p| \geq 1} \left| \left(\int_{-R}^R \frac{\sin r}{r} dr \right) \left\langle f_+ (p), \frac{g_0 (p; |p|)}{4 |p|^2} + \frac{p \cdot (\nabla_q g_0) (p; |p|)}{2 |p|^2} \right\rangle \right| dp \\
&\leq C \int_{|p| \geq 1} \left| \left\langle f_+ (p), \frac{g_0 (p; |p|)}{4 |p|^2} + \frac{p \cdot (\nabla_q g_0) (p; |p|)}{2 |p|^2} \right\rangle \right| dp \\
&\leq C \|f_+\|_{L^2} \left(\|g_+\|_{L^2} + \|g_+\|_{\mathcal{H}^1} \right),
\end{aligned}$$

and

$$\begin{aligned}
&\int_{|p| \leq 1} \left| \left(\int_{-R}^{R|p|} \frac{\sin r}{r} dr \right) \left\langle f_+ (p), \frac{g_0 (p; |p|)}{4 |p|^2} + \frac{p \cdot (\nabla_q g_0) (p; |p|)}{2 |p|^2} \right\rangle \right| dp \\
&\leq C \int_{|p| \leq 1} \left| \left\langle f_+ (p), \frac{g_0 (p; |p|)}{4 |p|^2} + \frac{p \cdot (\nabla_q g_0) (p; |p|)}{2 |p|^2} \right\rangle \right| dp \\
&\leq C \|f_+\|_{L^\infty(|p| \leq 1)} \left(\|g_+\|_{L^\infty(|p| \leq 1)} + \|g_+\|_{\mathcal{H}^1(|p| \leq 1)} \right),
\end{aligned}$$

uniformly on R, it follows from the dominated convergence theorem and the equality $\int_{-\infty}^{\infty} \frac{\sin r}{r} dr = \pi$ that

$$\begin{aligned}
&8\pi^2 i \lim_{R \to \infty} \int_{-1}^1 \int_{\mathbb{R}^3} \frac{\sin (Rr)}{r} F (|p| - r) \left\langle f_+ (p), \frac{g_0 (p; |p|)}{4 |p|^2} + \frac{p \cdot (\nabla_q g_0) (p; |p|)}{2 |p|^2} \right\rangle dpdr \\
&= 8\pi^3 i \int_{\mathbb{R}^3} \left\langle f_+ (p), \frac{g_0 (p; |p|)}{4 |p|^2} + \frac{p \cdot (\nabla_q g_0) (p; |p|)}{2 |p|^2} \right\rangle dp.
\end{aligned} \tag{3.30}$$

Therefore, taking the limit, as $R\to\infty$, in (3.28), and using (3.29), (3.30), we obtain

$$\lim_{R\to\infty} J_2^{2,2}(R) = 8\pi^3 i \int\limits_{\mathbb{R}^3} \left\langle f_+(p), \frac{g_0(p;|p|)}{4|p|^2} + \frac{p\cdot(\nabla_q g_0)(p;|p|)}{2|p|^2} \right\rangle dp. \tag{3.31}$$

Let us consider now $J_3^{2,2}(R)$. Observe that for all $0<\varepsilon<\frac{1}{2}$

$$\int\limits_{-1}^{1} \int\limits_{|p|\geq\frac{|r|}{3}} \frac{1}{|p|}\left| F(|p|-r)\left\langle f_+(p), \frac{g_0(p;|p|)}{4|p|^2} - \frac{p\cdot(\nabla_q g_0)(p;|p|)}{2|p|^2}\right\rangle\right| dpdr$$
$$\leq C\int\limits_0^1 \frac{dr}{r^{1-\varepsilon}}\int\limits_{\mathbb{R}^3}\left|\left\langle f_+(p), \frac{g_0(p;|p|)}{4|p|^{2+\varepsilon}} - \frac{p\cdot(\nabla_q g_0)(p;|p|)}{2|p|^{2+\varepsilon}}\right\rangle\right| dp$$
$$\leq C\|f_+\|_{L^\infty(|p|\leq 1)}\left(\|g_+\|_{L^\infty(|p|\leq1)} + \|g_+\|_{\mathcal{H}^1(|p|\leq 1)}\right) + C\|f_+\|_{L^2}\left(\|g_+\|_{L^2} + \|g_+\|_{\mathcal{H}^1}\right).$$

Then, arguing as in (3.10) we obtain

$$8\pi^2 i\lim_{R\to\infty}\int\limits_{-1}^{1}\sin(Rr)\int\limits_{|p|\geq\frac{|r|}{3}}\frac{1}{|p|}F(|p|-r)\left\langle f_+(p), \frac{g_0(p;|p|)}{4|p|^2} - \frac{p\cdot(\nabla_q g_0)(p;|p|)}{2|p|^2}\right\rangle dpdr = 0. \tag{3.32}$$

Note now that for $0<\varepsilon<1$

$$\left| w\left(|p|, \tfrac{p}{|p|}, r\right) F(|p|-r)\right| \leq C\frac{|r|^{\frac{3}{2}}}{|p|^{5/2}}|g_0(p;|p|)| + C\frac{1}{|p|^2}\left|\frac{p\cdot(\nabla_q g_0)(p;|p|)}{|p|}\right|$$
$$+C\frac{|r|^{\frac{3}{2}-\varepsilon}}{|p|^{\varepsilon}(|p|-r)^{2-2\varepsilon}}F(|p|-r)\left(\int\limits_0^{|p|}\left|\partial_t^2 g_0\left(t\tfrac{p}{|p|};|p|\right)\right|^2 t^2 dt\right)^{1/2}.$$

Then, for $\varepsilon<\frac{1}{2}$,

$$\int\limits_{-1}^{1}\int\limits_{|p|\geq\frac{|r|}{3}}\frac{1}{|p|\,r^2}\left|\left\langle f_+(p), w\left(|p|, \tfrac{p}{|p|}, r\right)\right\rangle F(|p|-r)(|p|-r)\right| dpdr$$
$$\leq C\int\limits_{\mathbb{R}^3}\frac{1}{|p|^{5/2}}|f_+(p)||g_0(p;|p|)|dp + C\int\limits_{-1}^{1}\frac{dr}{|r|^{1-\varepsilon}}\int\limits_{\mathbb{R}^3}\frac{|f_+(p)|}{|p|^{1+\varepsilon}}\left|\frac{p\cdot(\nabla_q g_0)(p;|p|)}{|p|}\right| dp$$
$$+C\int\limits_{-1}^{1}\frac{1}{|r|^{\frac{1}{2}+\varepsilon}}\int\limits_{|p|\geq\frac{|r|}{3}}\frac{|f_+(p)|}{|p|}\frac{F(|p|-r)}{(|p|-r)^{1-2\varepsilon}}\left(\int\limits_0^{|p|}\left|\partial_t^2 g_0\left(t\tfrac{p}{|p|};|p|\right)\right|^2 t^2 dt\right)^{1/2} dpdr.$$

Thus, using that

$$\int\limits_{-1}^{1}\frac{1}{|r|^{\frac{1}{2}+\varepsilon}}\int\limits_{|p|\geq\frac{|r|}{3}}\frac{|f_+(p)|}{|p|}\frac{F(|p|-r)}{(|p|-r)^{1-2\varepsilon}}\left(\int\limits_0^{|p|}\left|\partial_t^2 g_0\left(t\tfrac{p}{|p|};|p|\right)\right|^2 t^2 dt\right)^{1/2} dpdr$$
$$\leq C\int\limits_{|p|\leq 1}\left(\frac{1}{|p|^{2-2\varepsilon}} + \frac{1}{|p|^{\frac{3}{2}+\varepsilon}}\right)|f_+(p)|\left(\int\limits_0^1\left|\partial_t^2 g_0\left(t\tfrac{p}{|p|};|p|\right)\right|^2 t^2 dt\right)^{1/2} dp$$
$$+C\int\limits_{|p|\geq 1}\frac{|f_+(p)|}{|p|}\left(\left(\int\limits_0^1\left|\partial_t^2 g_0\left(t\tfrac{p}{|p|};|p|\right)\right|^2 t^2 dt\right)^{1/2} + \left(\int\limits_1^\infty\left|\partial_t^2 g_0\left(t\tfrac{p}{|p|};|p|\right)\right|^2 dt\right)^{1/2}\right) dp$$
$$\leq C\left(\|f_+\|_{L^\infty(|p|\leq 1)} + \|f_+\|_{L^2_{1/2+\varepsilon}}\right)\|g_+\|_{\mathcal{H}^2},$$

we obtain

$$\int_{-1}^{1}\int_{|p|\geq\frac{|r|}{3}}\frac{1}{|p|\,r^{2}}\left|\left\langle f_{+}(p),w\left(|p|,\tfrac{p}{|p|},r\right)\right\rangle F(|p|-r)(|p|-r)\right|dpdr$$
$$\leq C\left(\|f_{+}\|_{L^{\infty}(|p|\leq1)}\|g_{+}\|_{L^{\infty}(|p|\leq1)}+\|f_{+}\|_{L^{2}}\|g_{+}\|_{L^{2}}\right)$$
$$+C\left(\|f_{+}\|_{L^{\infty}(|p|\leq1)}+\|f_{+}\|_{L^{2}}\right)\|g_{+}\|_{\mathcal{H}^{1}}+C\left(\|f_{+}\|_{L^{\infty}(|p|\leq1)}+\|f_{+}\|_{L^{2}_{1/2+\varepsilon}}\right)\|g_{+}\|_{\mathcal{H}^{2}}$$
$$\leq C\left(\|f_{+}\|_{L^{\infty}(|p|\leq1)}+\|f_{+}\|_{L^{2}_{1/2+\varepsilon}}\right)\left(\|g_{+}\|_{L^{\infty}(|p|\leq1)}+\|g_{+}\|_{\mathcal{H}^{2}}\right).$$

Hence, arguing as in (3.10) we see that

$$-8\pi^{2}i\lim_{R\to\infty}\int_{-1}^{1}\sin(Rr)\int_{|p|\geq\frac{|r|}{3}}\frac{1}{|p|\,r^{2}}\left\langle f_{+}(p),w\left(|p|,\frac{p}{|p|},r\right)\right\rangle F(|p|-r)(|p|-r)\,dpdr=0. \tag{3.33}$$

Taking the limit, as $R\to\infty$, in (3.26), and taking in account (3.32) and (3.33) we arrive to

$$\lim_{R\to\infty}J_{3}^{2,2}(R)=0. \tag{3.34}$$

Moreover, passing to the limit, as $R\to\infty$, in (3.25), and using (3.27), (3.31) and (3.34) we get

$$-8\pi^{2}i\lim_{R\to\infty}\lim_{\delta\to0}\left[\int_{-1}^{-\delta}+\int_{\delta}^{1}\right]\frac{\sin(Rr)}{r^{2}}\int_{|p|\geq\frac{|r|}{3}}\left(\left\langle f_{+}(p),\frac{g_{0}\left((|p|-r)\frac{p}{|p|};|p|\right)}{2|p|-r}\right\rangle F(|p|-r)(|p|-r)\frac{dp}{|p|}\right)dr$$
$$=8\pi^{3}i\int_{\mathbb{R}^{3}}\left\langle f_{+}(p),\frac{g_{0}(p;|p|)}{4|p|^{2}}+\frac{p\cdot(\nabla_{q}g_{0})(p;|p|)}{2|p|^{2}}\right\rangle dp.$$

Using the last relation together with (3.24) in (3.23) we obtain

$$I_{1,2}^{2,2}(R)=8\pi^{3}i\int_{\mathbb{R}^{3}}\left\langle f_{+}(p),\frac{g_{0}(p;|p|)}{4|p|^{2}}+\frac{p\cdot(\nabla_{q}g_{0})(p;|p|)}{2|p|^{2}}\right\rangle dp+o(1), \tag{3.35}$$

as $R\to\infty$. Moreover, using equalities (3.22) and (3.35) in (3.21) we arrive to (3.20). ■

Now we show that $I_{1,2}^{3}(R)$ is $o(1)$ as $R\to\infty$. We have

Lemma 3.6 *Suppose that $f,g\in\mathcal{H}_{2}^{3/2+\varepsilon}$, $\varepsilon>0$. Then,*

$$\lim_{R\to\infty}I_{1,2}^{3}(R)=0.$$

Proof. Recall that

$$I_{1,2}^{3}(R)=-4\pi i\lim_{\delta\to0}\int_{\mathbb{R}^{3}}\left[\int_{-\infty}^{|p|-\delta}+\int_{|p|+\delta}^{\infty}\right]$$
$$\times\frac{\left(\sqrt{|p|^{2}+m^{2}}+\sqrt{r^{2}+m^{2}}\right)}{r|p|(|p|-r)(|p|+r)}\left\langle f_{+}(p),\int_{0}^{2\pi}\int_{0}^{\pi}\frac{\sin\left(R\sqrt{|p|^{2}-2r|p|\cos\theta+r^{2}}\right)}{\sqrt{|p|^{2}-2r|p|\cos\theta+r^{2}}}\partial_{\theta}g_{+}(r\omega(\theta,\varphi))\,d\theta d\varphi\right\rangle F(r)\,r^{2}drdp.$$

Noting that for all $0<\delta\leq1$

$$\left[\int_{|p|-1}^{|p|-\delta}+\int_{|p|+\delta}^{|p|+1}\right]\frac{\sqrt{|p|^{2}+m^{2}}}{(|p|-r)\,|p|}\left\langle f_{+}(p),\int_{0}^{2\pi}\int_{0}^{\pi}\frac{\sin\left(R|p|\sqrt{2-2\cos\theta}\right)}{|p|\sqrt{2-2\cos\theta}}\partial_{\theta}g_{+}(|p|\,\omega(\theta,\varphi))\,d\theta d\varphi\right\rangle dr=0,$$

we decompose $I_{1,2}^{3}(R)$ as follows

$$I_{1,2}^{3}(R)=I_{1,2}^{3,1}(R;0,\pi/4)+I_{1,2}^{3,1}(R;\pi/4,3\pi/4)+I_{1,2}^{3,1}(R;3\pi/4,\pi)+I_{1,2}^{3,2}(R;0,\pi/4)$$
$$+I_{1,2}^{3,2}(R;\pi/4,3\pi/4)+I_{1,2}^{3,2}(R;3\pi/4,\pi)+I_{1,2}^{3,3}(R)+I_{1,2}^{3,4}(R),$$

where

$$I_{1,2}^{3,1}(R;a,b) := -4\pi i \lim_{\delta\to 0}\int_{\mathbb{R}^3}\left[\int_{|p|-1}^{|p|-\delta} + \int_{|p|+\delta}^{|p|+1}\right]$$

$$\times \frac{1}{(|p|-r)}\left(\frac{\left(\sqrt{|p|^2+m^2}+\sqrt{r^2+m^2}\right)}{|p|(|p|+r)}F(r)\,r - \frac{\sqrt{|p|^2+m^2}}{|p|}\right)\left\langle f_+(p), i_{1,2}^{3,1}(|p|,r,\theta,\varphi;R;a,b)\,d\theta d\varphi\right\rangle drdp,$$

with

$$i_{1,2}^{3,1}(|p|,r;R;a,b) := \int_0^{2\pi}\int_a^b\left(\frac{\sin\left(R\sqrt{|p|^2-2r|p|\cos\theta+r^2}\right)}{\sqrt{|p|^2-2r|p|\cos\theta+r^2}}\partial_\theta g_+(r\omega(\theta,\varphi))\right)d\theta d\varphi, \tag{3.36}$$

$$I_{1,2}^{3,2}(R;a,b) := -4\pi i \lim_{\delta\to 0}\int_{\mathbb{R}^3}\left[\int_{|p|-1}^{|p|-\delta} + \int_{|p|+\delta}^{|p|+1}\right]\left(\frac{\sqrt{|p|^2+m^2}}{(|p|-r)|p|}\left\langle f_+(p), i_{1,2}^{3,2}(|p|,r,\theta,\varphi;R;a,b)\right\rangle\right)drdp,$$

with

$$i_{1,2}^{3,2}(|p|,r;R;a,b) := \int_0^{2\pi}\int_a^b\left(\frac{\sin\left(R\sqrt{|p|^2-2r|p|\cos\theta+r^2}\right)}{\sqrt{|p|^2-2r|p|\cos\theta+r^2}} - \frac{\sin\left(R|p|\sqrt{2-2\cos\theta}\right)}{|p|\sqrt{2-2\cos\theta}}\right)\partial_\theta g_+(r\omega(\theta,\varphi))\,d\theta d\varphi,$$

$$I_{1,2}^{3,3}(R) := -4\pi i \lim_{\delta\to 0}\int_{\mathbb{R}^3}\left[\int_{|p|-1}^{|p|-\delta} + \int_{|p|+\delta}^{|p|+1}\right]\left(\frac{\sqrt{|p|^2+m^2}}{(|p|-r)|p|}\left\langle f_+(p), i_{1,2}^{3,3}(|p|,r,\theta,\varphi;R)\right\rangle\right)drdp,$$

with

$$i_{1,2}^{3,3}(|p|,r;R) := \int_0^{2\pi}\int_0^{\pi b}\left(\frac{\sin\left(R|p|\sqrt{2-2\cos\theta}\right)}{|p|\sqrt{2-2\cos\theta}}\left(\partial_\theta g_+(r\omega(\theta,\varphi)) - \partial_\theta g_+(|p|\,\omega(\theta,\varphi))\right)\right)d\theta d\varphi,$$

and

$$I_{1,2}^{3,4}(R) := -4\pi i\int_{\mathbb{R}^3}\left[\int_{-\infty}^{|p|-1} + \int_{|p|+1}^{\infty}\right]\frac{\left(\sqrt{|p|^2+m^2}+\sqrt{r^2+m^2}\right)}{r|p|(|p|-r)(|p|+r)}$$

$$\times\left\langle f_+(p), \int_0^{2\pi}\int_0^{\pi}\frac{\sin\left(R\sqrt{|p|^2-2r|p|\cos\theta+r^2}\right)}{\sqrt{|p|^2-2r|p|\cos\theta+r^2}}\partial_\theta g_+(\omega(\theta,\varphi))\,d\theta d\varphi\right\rangle F(r)\,r^2 drdp.$$

Let us consider first $I_{1,2}^{3,1}(R;0,\pi/4)$. Using that $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$ we have

$$\frac{\sin\left(R\sqrt{|p|^2-2r|p|\cos\theta+r^2}\right)}{\sqrt{|p|^2-2r|p|\cos\theta+r^2}}\partial_\theta g_+(r\omega(\theta,\varphi)) = \frac{e^{iR\sqrt{|p|^2-2r|p|\cos\theta+r^2}} - e^{-iR\sqrt{|p|^2-2r|p|\cos\theta+r^2}}}{2i\sqrt{|p|^2-2r|p|\cos\theta+r^2}}\partial_\theta g_+(r\omega(\theta,\varphi)).$$

Noting that

$$\frac{e^{\pm iR\sqrt{|p|^2-2r|p|\cos\theta+r^2}}}{\sqrt{|p|^2-2r|p|\cos\theta+r^2}} = \frac{\partial_\theta\left((\sin\theta)\,e^{\pm iR\sqrt{|p|^2-2r|p|\cos\theta+r^2}}\right)}{(\cos\theta)\sqrt{|p|^2-2r|p|\cos\theta+r^2}\pm iRr|p|\sin^2\theta}$$

and integrating by parts we get

$$\begin{aligned}\int_0^{\pi/4}\frac{e^{\pm iR\sqrt{|p|^2-2r|p|\cos\theta+r^2}}}{\sqrt{|p|^2-2r|p|\cos\theta+r^2}}\partial_\theta g_+(r\omega(\theta,\varphi))\,d\theta &= \int_0^{\pi/4}\frac{\partial_\theta\left((\sin\theta)e^{\pm iR\sqrt{|p|^2-2r|p|\cos\theta+r^2}}\right)}{\cos\theta\sqrt{|p|^2-2r|p|\cos\theta+r^2}\pm iRr|p|\sin^2\theta}\partial_\theta g_+(r\omega(\theta,\varphi))\,d\theta\\ &= \sqrt{2}\frac{e^{\pm iR\sqrt{|p|^2-\sqrt{2}r|p|+r^2}}}{\sqrt{2}\sqrt{|p|^2-\sqrt{2}r|p|+r^2}\pm iRr|p|}\;\partial_\theta g_+(r\omega(\theta,\varphi))\big|_{\theta=\pi/4}\\ &\quad -\int_0^{\pi/4}e^{\pm iR\sqrt{|p|^2-2r|p|\cos\theta+r^2}}\sin\theta\left(\partial_\theta\frac{\partial_\theta g_+(r\omega(\theta,\varphi))}{\cos\theta\sqrt{|p|^2-2r|p|\cos\theta+r^2}\pm iRr|p|\sin^2\theta}\right)d\theta,\end{aligned}$$

and hence,

$$i_{1,2}^{3,1}\left(|p|,r;R;0,\pi/4\right)=j_{1,2}^{3,1}\left(|p|,r;R\right)+j_{1,2}^{3,1}\left(|p|,r;-R\right)+l_{1,2}^{3,1}\left(|p|,r;R\right)+l_{1,2}^{3,1}\left(|p|,r;-R\right), \tag{3.37}$$

with

$$j_{1,2}^{3,1}\left(|p|,r;\pm R\right):=\pm\frac{\sqrt{2}}{2i}\left(\frac{e^{\pm iR\sqrt{|p|^2-\sqrt{2}r|p|+r^2}}}{\sqrt{2}\sqrt{|p|^2-\sqrt{2}r\,|p|+r^2}\pm iRr\,|p|}\right)\int_0^{2\pi}\partial_\theta g_+\left(r\omega(\theta,\varphi)\right)\big|_{\theta=\pi/4}\,d\varphi$$

and

$$l_{1,2}^{3,1}\left(|p|,r;\pm R\right):=\mp\frac{1}{2i}\int_0^{2\pi}\int_0^{\pi/4}e^{\pm iR\sqrt{|p|^2-2r|p|\cos\theta+r^2}}\sin\theta\left(\partial_\theta\frac{\partial_\theta g_+\left(r\omega(\theta,\varphi)\right)}{\cos\theta\sqrt{|p|^2-2r\,|p|\cos\theta+r^2}\pm iRr\,|p|\sin^2\theta}\right)d\theta d\varphi. \tag{3.38}$$

Since

$$\left|\int_0^{2\pi}\partial_\theta g_+\left(r\omega(\theta,\varphi)\right)d\varphi\right|=\left|\int_0^{2\pi}\partial_\theta g_+\left(r\omega(\theta,\varphi)\right)-\,\partial_\theta g_+\left(r\omega(\theta,\varphi)\right)\big|_{\theta=0}\,d\varphi\right|$$
$$\leq C\int_0^{2\pi}\left|\int_0^{\theta}\partial_\theta^2 g_+\left(r\omega(\theta,\varphi)\right)d\theta\right|d\varphi\leq C\int_0^{2\pi}\int_0^{\pi}\left|\partial_\theta^2 g_+\left(r\omega(\theta,\varphi)\right)\right|d\theta d\varphi$$

and

$$\left|\frac{e^{\pm iR\sqrt{|p|^2-\sqrt{2}r|p|+r^2}}}{\sqrt{2}\sqrt{|p|^2-\sqrt{2}r\,|p|+r^2}\pm iRr\,|p|}\right|\leq\frac{1}{\left||p|-r\right|^{1-\beta}\left(Rr\,|p|\right)^{\beta}},\quad\beta\leq 1,$$

we get

$$\left|j_{1,2}^{3,1}\left(|p|,r;\pm R\right)\right|\leq C\frac{1}{\left||p|-r\right|^{1-\beta}\left(Rr\,|p|\right)^{\beta}}\int_0^{2\pi}\int_0^{\pi/4}\left|\partial_\theta^2 g_+\left(r\omega(\theta,\varphi)\right)\right|d\theta d\varphi. \tag{3.39}$$

Note that the following estimates are true

$$\cos^2\theta\left(|p|^2-2r\,|p|\cos\theta+r^2\right)+\left(Rr\,|p|\right)^2\sin^4\theta\geq C\left(|p|-r\right)^{2-2\beta}\left(Rr\,|p|\right)^{2\beta}\sin^{4\beta}\theta, \tag{3.40}$$

for $\beta\leq 1$ and $\theta\in[0,\pi/4]$,

$$\frac{|r|\,|p|\sin\theta\cos\theta}{\sqrt{|p|^2-2r\,|p|\cos\theta+r^2}}\leq C\sqrt{r\,|p|} \tag{3.41}$$

and

$$\left|\int_0^{2\pi}\partial_\theta g_+\left(r\omega(\theta,\varphi)\right)d\varphi\right|=\left|\int_0^{2\pi}\left(\partial_\theta g_+\left(r\omega(\theta,\varphi)\right)-\,\partial_\theta g_+\left(r\omega(\theta,\varphi)\right)\big|_{\theta=0}\right)d\varphi\right|\leq C\,|\theta|^{1/2}A\left(g_+;r\right), \tag{3.42}$$

where

$$A\left(g_+;r\right):=\left(\int_0^{2\pi}\int_0^{\pi}\left|\partial_\theta^2 g_+\left(r\omega(\theta,\varphi)\right)\right|^2d\theta d\varphi\right)^{1/2}.$$

Using (3.40) with $\beta=3/4-\varepsilon$, $\varepsilon>0$, we get

$$\begin{aligned}&\left|\int_0^{2\pi}\int_0^{\pi/4}\sin\theta\left(\frac{\partial_\theta^2 g_+\left(r\omega(\theta,\varphi)\right)}{\cos\theta\sqrt{|p|^2-2r\,|p|\cos\theta+r^2}\pm iRr\,|p|\sin^2\theta}\right)d\theta d\varphi\right|\\&\leq\frac{C}{\left||p|-r\right|^{1-\beta}\left(Rr\,|p|\right)^{\beta}}\int_0^{2\pi}\int_0^{\pi/4}\left(\frac{\left|\partial_\theta^2 g_+\left(r\omega(\theta,\varphi)\right)\right|}{\left(\sin\theta\right)^{2\beta-1}}\right)d\theta d\varphi\leq\frac{C}{\left||p|-r\right|^{1/4+\varepsilon}\left(Rr\,|p|\right)^{3/4-\varepsilon}}A\left(g_+;r\right).\end{aligned} \tag{3.43}$$

It follows from (3.40), (3.41) and the estimate

$$\frac{\sqrt{|p|^2-2r\,|p|\cos\theta+r^2}\sin\theta}{\cos^2\theta\left(|p|^2-2r\,|p|\cos\theta+r^2\right)+\left(Rr\,|p|\right)^2\sin^4\theta}\leq\frac{\sin\theta}{\left(\cos^2\theta\left(|p|^2-2r\,|p|\cos\theta+r^2\right)+\left(Rr\,|p|\right)^2\sin^4\theta\right)^{1/2}}$$

that

$$\left|\partial_\theta \frac{1}{\cos\theta\sqrt{|p|^2-2r|p|\cos\theta+r^2}+iRr|p|\sin^2\theta}\right| \le C\frac{\sqrt{|p|^2-2r|p|\cos\theta+r^2}\sin\theta+\sqrt{r|p|}+Rr|p|\sin\theta}{\cos^2\theta\left(|p|^2-2r|p|\cos\theta+r^2\right)+(Rr|p|)^2\sin^4\theta}$$
$$\le C\frac{\sin\theta}{(|p|-r)^{1-\beta_1}(Rr|p|)^{\beta_1}(\sin\theta)^{2\beta_1}}+C\frac{\sqrt{r|p|}}{(|p|-r)^{2-2\beta_2}(Rr|p|)^{2\beta_2}(\sin\theta)^{4\beta_2}}$$
$$+C\frac{Rr|p|\sin\theta}{(|p|-r)^{2-2\beta_3}(Rr|p|)^{2\beta_3}(\sin\theta)^{4\beta_3}},$$

for $\beta_j\le 1$, $j=1,2,3$. Thus, using (3.42) we have

$$\left|\int_0^{2\pi}\int_0^{\pi/4}\sin\theta\left(\partial_\theta\frac{1}{\cos\theta\sqrt{|p|^2-2r|p|\cos\theta+r^2}+iRr|p|\sin^2\theta}\right)\partial_\theta g_+(r\omega(\theta,\varphi))\,d\theta d\varphi\right|$$
$$\le C\left(\left(\frac{1}{||p|-r|^{1-\beta_1}(Rr|p|)^{\beta_1}}\int_0^{\pi/4}\frac{d\theta}{(\sin\theta)^{2\beta_1-5/2}}+\frac{\sqrt{r|p|}}{||p|-r|^{2-2\beta_2}(Rr|p|)^{2\beta_2}}\int_0^{\pi/4}\frac{d\theta}{(\sin\theta)^{4\beta_2-3/2}}\right)A(g_+;r)\right) \tag{3.44}$$
$$+C\left(\frac{1}{||p|-r|^{2-2\beta_3}(Rr|p|)^{2\beta_3-1}}\int_0^{\pi/4}\frac{d\theta}{(\sin\theta)^{4\beta_3-5/2}}\right)A(g_+;r).$$

Moreover, taking $\beta_1=3/4-\varepsilon$, $\beta_2=5/8-(1/2)\varepsilon$ and $\beta_3=7/8-(1/2)\varepsilon$, for $\varepsilon>0$, in (3.44), and using the resulting estimate, together with (3.43) in (3.38), we get

$$\left|l_{1,2}^{3,1}(|p|,r;\pm R)\right|\le C\left(\frac{1}{||p|-r|^{1/4+\varepsilon}(Rr|p|)^{3/4-\varepsilon}}+\frac{\sqrt{r|p|}}{||p|-r|^{3/4+\varepsilon}(Rr|p|)^{5/4-\varepsilon}}\right)A(g_+;r). \tag{3.45}$$

Using (3.39), with $\beta=3/4-\varepsilon$, and (3.45) in (3.37) we get

$$\begin{aligned}\left|i_{1,2}^{3,1}(|p|,r;R;0,\pi/4)\right|&\le C\left(\tfrac{1}{||p|-r|^{1/4+\varepsilon}(Rr|p|)^{3/4-\varepsilon}}+\tfrac{\sqrt{r|p|}}{||p|-r|^{3/4+\varepsilon}(Rr|p|)^{5/4-\varepsilon}}\right)A(g_+;r)\\ &\le C\left(\tfrac{1}{||p|-r|^{1/4+\varepsilon}(Rr|p|)^{3/4-\varepsilon}}+\tfrac{\sqrt{r|p|}}{||p|-r|^{3/4+\varepsilon}(Rr|p|)^{5/4-\varepsilon}}\right)(B(g_+;r))^{1/2},\end{aligned} \tag{3.46}$$

with

$$B(g_+;r):=\int_0^{2\pi}\int_0^{\pi}\left(|\partial_\theta g_+(r\omega(\theta,\varphi))|^2+\left|\partial_\theta^2 g_+(r\omega(\theta,\varphi))\right|^2\right)d\theta d\varphi. \tag{3.47}$$

On the other hand, by (3.36),

$$\left|i_{1,2}^{3,1}(|p|,r;R;0,\pi/4)\right|\le CR\left(\int_0^{2\pi}\int_0^{\pi}|\partial_\theta g_+(r\omega(\theta,\varphi))|^2\,d\theta d\varphi\right)^{1/2}\le CR(B(g_+;r))^{1/2}. \tag{3.48}$$

Thus, observing that

$$\left|\frac{\left(\sqrt{|p|^2+m^2}+\sqrt{r^2+m^2}\right)}{|p|(|p|+r)}F(r)\,r-\frac{\sqrt{|p|^2+m^2}}{|p|}\right|\le$$
$$\le\left|\left(\frac{\left(\sqrt{|p|^2+m^2}+\sqrt{r^2+m^2}\right)}{|p|(|p|+r)}-\frac{\sqrt{|p|^2+m^2}}{|p|^2}\right)r\right|+\frac{\sqrt{|p|^2+m^2}}{|p|^2}\left|(F(r)\,r-F(|p|)\,|p|)\right|\le C\left(\langle p\rangle^{-1}+\tfrac{1}{|p|^2}\right)||p|-r|,$$

we obtain by using (3.46) and (3.48)

$$\begin{aligned}
&\int_{\mathbb{R}^3}\int_{|p|-1}^{|p|+1}\frac{1}{||p|-r|}\left|\frac{\left(\sqrt{|p|^2+m^2}+\sqrt{r^2+m^2}\right)}{|p|(|p|+r)}F(r)\,r-\frac{\sqrt{|p|^2+m^2}}{|p|}\right||f_+(p)|\left|i_{1,2}^{3,1}(|p|,r;R;0,\pi/4)\right|drdp\leq\\
&\leq C\int_{\mathbb{R}^3}\left(\langle p\rangle^{-1}+\tfrac{1}{|p|^2}\right)|f_+(p)|\int_{|p|-1}^{|p|+1}\left|i_{1,2}^{3,1}(|p|,r;R;0,\pi/4)\right|^{\alpha}\left|i_{1,2}^{3,1}(|p|,r;R;0,\pi/4)\right|^{1-\alpha}drdp\\
&\leq CR^{1-\alpha}\int_{\mathbb{R}^3}\left(\langle p\rangle^{-1}+\tfrac{1}{|p|^2}\right)\left(\int_{|p|-1}^{|p|+1}B(g_+;r)\,dr\right)^{1/2}\\
&\times|f_+(p)|\left(\int_{|p|-1}^{|p|+1}\left(\tfrac{1}{||p|-r|^{1/4+\varepsilon}(Rr|p|)^{3/4-\varepsilon}}\right)^{2\alpha}+\left(\tfrac{\sqrt{r|p|}}{||p|-r|^{3/4+\varepsilon}(Rr|p|)^{5/4-\varepsilon}}\right)^{2\alpha}dr\right)^{1/2},
\end{aligned}\tag{3.49}$$

for $\alpha\leq 1$. Note that

$$\begin{aligned}
&\int_{|p|-1}^{|p|+1}\left(\tfrac{1}{||p|-r|^{1/4+\varepsilon}(Rr|p|)^{3/4-\varepsilon}}\right)^{2\alpha}+\left(\tfrac{\sqrt{r|p|}}{||p|-r|^{3/4+\varepsilon}(Rr|p|)^{5/4-\varepsilon}}\right)^{2\alpha}dr\\
&\leq R^{-2\alpha(5/4-\varepsilon)}|p|^{-2\alpha(3/4-\varepsilon)}\int_{|p|-1}^{|p|+1}\left(\tfrac{1}{||p|-r|^{3/4+\varepsilon}|r|^{3/4-\varepsilon}}\right)^{2\alpha}dr+(R|p|)^{-2\alpha(3/4-\varepsilon)}\int_{|p|-1}^{|p|+1}\left(\tfrac{1}{||p|-r|^{1/4+\varepsilon}|r|^{3/4-\varepsilon}}\right)^{2\alpha}dr\\
&\leq CR^{-2\alpha(3/4-\varepsilon)}\left(1+\tfrac{1}{|p|^{4\alpha(3/4+\varepsilon)}}\right).
\end{aligned}$$

Therefore, taking $\alpha=4/7+\varepsilon$ in (3.49) and using the last inequality and (3.47), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^3}\int_{|p|-1}^{|p|+1}\frac{1}{||p|-r|}\left|\frac{\left(\sqrt{|p|^2+m^2}+\sqrt{r^2+m^2}\right)}{|p|(|p|+r)}F(r)\,r-\frac{\sqrt{|p|^2+m^2}}{|p|}\right||f_+(p)|\left|i_{1,2}^{3,1}(|p|,r;R;0,\pi/4)\right|drdp\\
&\leq\frac{C}{R^{\varepsilon_1}}\left(\int_{\mathbb{R}^3}\left(1+\tfrac{1}{|p|^2}\right)|f_+(p)|\left(1+\tfrac{1}{|p|^{1-\varepsilon_1}}\right)dp\right)\|g_+\|_{\mathcal{H}^2}\leq\frac{C}{R^{\varepsilon_1}}\left(\left(\|f_+\|_{L^\infty}+\|f_+\|_{L^2}\right)\|g_+\|_{\mathcal{H}^2}\right),
\end{aligned}$$

for $\varepsilon_1>0$ small enough, and furthermore, we conclude that

$$\lim_{R\to\infty}I_{1,2}^{3,1}(R;0,\pi/4)=0.$$

Proceeding similarly, we obtain

$$\lim_{R\to\infty}I_{1,2}^{3,1}(R;\pi/4,3\pi/4)=0,$$

$$\lim_{R\to\infty}I_{1,2}^{3,1}(R;3\pi/4,\pi)=0,$$

$$\lim_{R\to\infty}I_{1,2}^{3,2}(R;0,\pi/4)=0,$$

$$\lim_{R\to\infty}I_{1,2}^{3,2}(R;\pi/4,3\pi/4)=0$$

and

$$\lim_{R\to\infty}I_{1,2}^{3,2}(R;3\pi/4,\pi)=0.$$

Actually, the proof for the parts $I_{1,2}^{3,1}(R;\pi/4,3\pi/4)$ and $I_{1,2}^{3,2}(R;\pi/4,3\pi/4)$ is more simple since $\sin\theta\neq 0$ on $[\pi/4,3\pi/4]$.

Let us now consider $I_{1,2}^{3,2}(R)$. Note that for $\alpha \leq 1$

$$\left| \int_0^{2\pi} \left(\partial_\theta g_+ \left(r\omega(\theta,\varphi) \right) - \partial_\theta g_+ \left(|p| \, \omega(\theta,\varphi) \right) \right) d\varphi \right|$$

$$= \left| \int_0^{2\pi} \left(\partial_\theta g_+ \left(r\omega(\theta,\varphi) \right) - \partial_\theta g_+ \left(|p| \, \omega(\theta,\varphi) \right) \right) d\varphi \right|^{\alpha} \left| \int_0^{2\pi} \left(\partial_\theta g_+ \left(r\omega(\theta,\varphi) \right) - \partial_\theta g_+ \left(|p| \, \omega(\theta,\varphi) \right) \right) d\varphi \right|^{1-\alpha}$$

$$\leq C \left| |p| - r \right|^{\alpha/2} |\theta|^{(1/2)(1-\alpha)} \left(\int_0^{2\pi} \int_{|p|}^{r} \left| \partial_r \partial_\theta g_+ \left(r\omega(\theta,\varphi) \right) \right|^2 dr d\varphi \right)^{\alpha/2} \left(\left(A\left(g_+; r \right) \right)^{\frac{1-\alpha}{2}} + A\left(g_+; |p| \right)^{\frac{1-\alpha}{2}} \right).$$

Then, by Hölder inequality we get for $\alpha < 1/2$

$$\left| i_{1,2}^{3,2}\left(|p|, r; R \right) \right| \leq \int_0^{\pi} \left(\frac{1}{\sqrt{2 - 2\cos\theta}} \left| \int_0^{2\pi} \left(\partial_\theta g_+ \left(r\omega(\theta,\varphi) \right) - \partial_\theta g_+ \left(|p| \, \omega(\theta,\varphi) \right) \right) d\varphi \right| \right) d\theta$$

$$\leq C \left| |p| - r \right|^{\alpha/2} \left(\left(A\left(g_+; r \right) \right)^{\frac{1-\alpha}{2}} + A\left(g_+; |p| \right)^{\frac{1-\alpha}{2}} \right) \int_0^{\pi/4} \left(|\theta|^{-1/2-(1/2)\alpha} \left(\int_0^{2\pi} \int_{|p|}^{r} \left| \partial_r \partial_\theta g_+ \left(r\omega(\theta,\varphi) \right) \right|^2 dr d\varphi \right)^{\alpha/2} \right) d\theta$$

$$\leq C \left| |p| - r \right|^{\alpha/2} \left(\left(A\left(g_+; r \right) \right)^{\frac{1-\alpha}{2}} + A\left(g_+; |p| \right)^{\frac{1-\alpha}{2}} \right) \left(\|g_+\|_{\mathcal{H}^2} \right)^{\alpha/2}.$$

Thus, we have

$$\int_{\mathbb{R}^3} \int_{|p|-1}^{|p|+1} \frac{\sqrt{|p|^2 + m^2}}{\left| |p| - r \right| |p|} \left| f_+(p) \right| \left| i_{1,2}^{3,2}\left(|p|, r, \theta, \varphi; R \right) \right| dr dp$$

$$\leq C \left(\|g_+\|_{\mathcal{H}^2} \right)^{\alpha/2} \int_{-\infty}^{\infty} \left(A\left(g_+; r \right) \right)^{\frac{1-\alpha}{2}} \int_{\mathbb{R}^3} \frac{\sqrt{|p|^2 + m^2}}{\left| |p| - r \right|^{1-\alpha/2} |p|} \left| f_+(p) \right| dp dr$$

$$+ C \left(\|g_+\|_{\mathcal{H}^2} \right)^{\alpha/2} \int_{\mathbb{R}^3} \frac{\sqrt{|p|^2 + m^2}}{|p|} \left| f_+(p) \right| A\left(g_+; |p| \right)^{\frac{1-\alpha}{2}} \left(\int_{|p|-1}^{|p|+1} \frac{1}{\left| |p| - r \right|^{1-\alpha/2}} dr \right) dp$$

$$\leq C \|f_+\|_{L_1^\infty} \|g_+\|_{\mathcal{H}^2_{3/2+\varepsilon}}.$$

Therefore, arguing as in (3.10) we get

$$\lim_{R \to \infty} I_{1,2}^{3,2}(R) = 0.$$

Finally, the proof for the term $I_{1,2}^{3,4}(R)$ is similar to that of $I_{1,2}^{3,1}$ or $I_{1,2}^{3,2}$. It results to be easier since there is no irregular term $\left| |p| - r \right|^{-1}$. ■

We already obtained the asymptotics of $I_1(R)$ and $I_4(R)$ as $R \to \infty$. Let us now study the behavior of $I_2(R)$ and $I_3(R)$ for big R. We prove the following

Lemma 3.7 *Let $f, g \in \mathcal{H}_2^{3/2+\varepsilon}$, $\varepsilon > 0$. Then,*

$$\lim_{R \to \infty} I_2(R) = \lim_{R \to \infty} I_3(R) = 0.$$

Proof. We consider the term $I_2(R)$. The proof for $I_3(R)$ is analogous. Let $\varphi \in C_0^\infty(\mathbb{R})$, such that $\varphi(0) = 1$. Then, it follows from the proof of Lemma 3.1 (see relation (3.2)) and the dominated convergence theorem that

$$I_2(R) = \lim_{\varepsilon \to \infty} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left\langle \left(\int_0^\infty e^{-i\left(\sqrt{|p|^2+m^2} + \sqrt{|q|^2+m^2} \right) t} \varphi(\varepsilon t) \, dt \right) f_+(p), \tilde{\zeta}_R(p-q) \, g_-(q) \right\rangle dq dp$$

$$= -i \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left\langle f_+(p), \tilde{\zeta}_R(p-q) \frac{g_-(q)}{\sqrt{|p|^2+m^2} + \sqrt{|q|^2+m^2}} \right\rangle dq dp$$

$$- i \lim_{\varepsilon \to \infty} \varepsilon \int_0^\infty \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \varphi'(\varepsilon t) \left\langle \frac{e^{-i\left(\sqrt{|p|^2+m^2} + \sqrt{|q|^2+m^2} \right) t}}{\sqrt{|p|^2+m^2} + \sqrt{|q|^2+m^2}} f_+(p), \tilde{\zeta}_R(p-q) \, g_-(q) \right\rangle dq dp dt.$$

Integrating by parts, as in expression (3.4), in the second integral of the R.H.S. of the last relation, we show that the limit $-i\lim_{\varepsilon\to\infty}\int_0^\infty\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\varphi'(\varepsilon t)\left\langle\frac{e^{-i\left(\sqrt{|p|^2+m^2}+\sqrt{|q|^2+m^2}\right)t}}{\sqrt{|p|^2+m^2}+\sqrt{|q|^2+m^2}}f_+(p),\tilde{\zeta}_R(p-q)\,g_-(q)\right\rangle dqdpdt$ exists, and hence,

$$I_2(R)=-i\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\left\langle f_+(p),\tilde{\zeta}_R(p-q)\frac{g_-(q)}{\sqrt{|p|^2+m^2}+\sqrt{|q|^2+m^2}}\right\rangle dqdp.$$

Passing to the spherical coordinate system, where the $z-$axis is directed along the vector p, we obtain

$$I_{1,2}(R)=-i\int_{\mathbb{R}^3}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty}\left\langle f_+(p),\tilde{\zeta}_R(p-r\omega(\theta,\varphi))\,g_1(r\omega(\theta,\varphi);|p|)\right\rangle r^2\sin\theta drd\theta d\varphi dp,$$

where $g_1(r\omega(\theta,\varphi);|p|):=\frac{g_-(r\omega(\theta,\varphi))}{\sqrt{|p|^2+m^2}+\sqrt{r^2+m^2}}$ and $\omega(\theta,\varphi)=(\cos\varphi\sin\theta,\sin\varphi\sin\theta,\cos\theta)$. Using (3.14) with g_1 instead of g_0 we have

$$\int_0^{\pi}\tilde{\zeta}_R(p-r\omega(\theta,\varphi))\,g_1(r\omega(\theta,\varphi);|p|)\sin\theta d\theta$$

$$=-\frac{\sin R(|p|+r)}{r|p|(|p|+r)}g_1\left(-r\frac{p}{|p|};|p|\right)+\frac{\sin R(|p|-r)}{r|p|(|p|-r)}g_1\left(r\frac{p}{|p|};|p|\right)+\frac{1}{r|p|}\int_0^{\pi}\frac{\sin R\sqrt{|p|^2-2r|p|\cos\theta+r^2}}{\sqrt{|p|^2-2r|p|\cos\theta+r^2}}\partial_\theta g_1(r\omega(\theta,\varphi);|p|)\,d\theta,$$

and hence,

$$I_2(R):=I_2^1(R)+I_2^2(R)+I_2^3(R),\tag{3.50}$$

with

$$I_2^1(R):=8\pi^2 i\int_{\mathbb{R}^3}\int_0^{\infty}\left\langle f_+(p),\frac{\sin R(|p|+r)}{|p|(|p|+r)}g_1\left(-r\frac{p}{|p|};|p|\right)\right\rangle rdrdp,$$

$$I_2^2(R):=-8\pi^2 i\int_{\mathbb{R}^3}\int_0^{\infty}\left\langle f_+(p),\frac{\sin R(|p|-r)}{|p|(|p|-r)}g_1\left(r\frac{p}{|p|};|p|\right)\right\rangle rdrdp,$$

and

$$I_2^3(R):=-4\pi i\int_{\mathbb{R}^3}\int_0^{\infty}\frac{\left\langle f_+(p),i_2^3(r,p;R)\right\rangle}{|p|\left(\sqrt{|p|^2+m^2}+\sqrt{r^2+m^2}\right)}rdrdp,$$

where

$$i_2^3(r,p;R):=\int_0^{2\pi}\int_0^{\pi}\frac{\sin R\sqrt{|p|^2-2r|p|\cos\theta+r^2}}{\sqrt{|p|^2-2r|p|\cos\theta+r^2}}\partial_\theta g_-(r\omega(\theta,\varphi))\,d\theta d\varphi.$$

Since

$$\int_{\mathbb{R}^3}\int_0^{\infty}|f_+(p)|\frac{\left|g_1\left(-r\frac{p}{|p|};|p|\right)\right|}{|p|(|p|+r)}rdrdp\leq\int_0^{\infty}\left(\int_{\mathbb{S}^2}|f_+(p)|^2d\omega\right)^{1/2}d|p|\int_0^{\infty}\left(\int_{\mathbb{S}^2}\frac{|g_-(r\omega)|^2}{r^2+m^2}d\omega\right)^{1/2}rdr$$
$$\leq C\left(\|f_+\|_{L^\infty(|p|\leq1)}+\|f_+\|_{L^2(\mathbb{R}^3)}\right)\left(\|g_-\|_{L^\infty(|p|\leq1)}+\|g_-\|_{L^2(\mathbb{R}^3)}\right),$$

arguing as in (3.10) we get

$$\lim_{R\to\infty}I_2^1(R)=0.\tag{3.51}$$

Similarly to Lemma 3.4 in the case of $I_{1,2}^1(R)$ we prove that

$$\lim_{R\to\infty}I_2^2(R)=0.\tag{3.52}$$

Observing that $\int_0^{2\pi}\partial_\theta g_-(r\omega(\theta,\varphi))|_{\theta=0}\,d\varphi=0$ we have

$$i_2^3(r,p;R)=\int_0^{2\pi}\int_0^{\pi}\frac{\sin R\sqrt{|p|^2-2r|p|\cos\theta+r^2}}{\sqrt{|p|^2-2r|p|\cos\theta+r^2}}\left(\partial_\theta g_-(r\omega(\theta,\varphi))-\partial_\theta g_-(r\omega(\theta,\varphi))|_{\theta=0}\right)d\theta d\varphi.$$

Then,

$$\left|i_2^3(r,p;R)\right| \leq \frac{C}{\sqrt{r|p|}} \int_0^{\pi} \frac{\theta^{1/2}}{\sqrt{1-\cos\theta}} d\theta \left(\int_0^{2\pi}\int_0^{\pi} \left|\partial_\theta^2 g_-(r\omega(\theta,\varphi))\right|^2 d\theta d\varphi \right)^{1/2} \leq \frac{C}{\sqrt{r|p|}} \left(\int_0^{2\pi}\int_0^{\pi} \left|\partial_\theta^2 g_-(r\omega(\theta,\varphi))\right|^2 d\theta d\varphi \right)^{1/2},$$

and moreover,

$$\int_{\mathbb{R}^3}\int_0^{\infty} \frac{|f_+(p)|\left|i_2^3(r,p;R)\right|}{|p|\left(\sqrt{|p|^2+m^2}+\sqrt{r^2+m^2}\right)} r dr dp \leq C \int_{\mathbb{R}^3} \frac{|f_+(p)|}{|p|^{3/2}} dp \left(\int_0^{\infty} \frac{1}{\sqrt{r^2+m^2}} \left(\int_0^{2\pi}\int_0^{\pi} \left|\partial_\theta\left(\omega(\theta,\varphi)\cdot\nabla g_-(r\omega(\theta,\varphi))\right)\right|^2 r^2 d\theta d\varphi \right)^{1/2} dr \right)$$
$$\leq C\left(\|f_+\|_{L^\infty(|p|\leq 1)} + \|f_+\|_{L_1^2(\mathbb{R}^3)} \right) \|g_+\|_{\mathcal{H}_1^2}.$$

Hence, arguing as in (3.10) we see that

$$\lim_{R\to\infty} I_2^3(R) = 0. \tag{3.53}$$

Taking the limit as $R\to\infty$ in (3.50) and using (3.51), (3.52) and (3.53) we conclude that

$$\lim_{R\to\infty} I_2(R) = 0.$$

■

4 Time delay.

4.1 Proof of Theorem 1.1.

We begin by presenting a result that allows us to express the time delay $\delta\mathcal{T}(f)$ in terms of the scattering operator $\mathbf{S}$ (see, for example, Proposition 7.22, page 365 of [6]).

Proposition 4.1 *Let f be such that, for every fixed $R<\infty$, each of the functions $t\to \left\|\zeta\left(\frac{|x|}{R}\right)e^{-iH_0t}f\right\|_{L^2(\mathbb{R}^3)}$ and $t\to \left\|\zeta\left(\frac{|x|}{R}\right)e^{-iH_0t}\mathbf{S}f\right\|_{L^2(\mathbb{R}^3)}$ belong to $L^2([0,\infty))$. Assume that the wave operators exist and are complete. Moreover, suppose that the function $t\to\left\|\left(W_- - e^{itH}e^{-itH_0}\right)f\right\|_{L^2(\mathbb{R}^3)}$ belongs to $L^1((-\infty,0])$ and that $t\to\left\|\left(W_+ - e^{itH}e^{-itH_0}\right)\mathbf{S}f\right\|_{L^2(\mathbb{R}^3)}$ belongs to $L^1([0,\infty))$. Then,*

$$\delta\mathcal{T}(f) = \lim_{R\to\infty}\int_0^{\infty}\left\{\left\|\zeta\left(\frac{|x|}{R}\right)e^{-iH_0t}\mathbf{S}f\right\|^2 - \left\|\zeta\left(\frac{|x|}{R}\right)e^{-iH_0t}f\right\|^2\right\}dt. \tag{4.1}$$

In particular, Proposition 4.1 relates the time delay with the expectation values $I(R)$, defined by (1.9). To prove Theorem 1.1 first we need to show that under the assumptions of Theorem 1.1 relation (4.1) holds. We only have to verify that the functions $t\to\left\|\zeta\left(\frac{|x|}{R}\right)e^{-iH_0t}f\right\|_{L^2(\mathbb{R}^3)}$ and $t\to\left\|\zeta\left(\frac{|x|}{R}\right)e^{-iH_0t}\mathbf{S}f\right\|_{L^2(\mathbb{R}^3)}$ belong to $L^2([0,\infty))$. Let us prove that this is true if $f\in\mathcal{H}_2^{3/2+\varepsilon}\left(\mathbb{R}^3;\mathbb{C}^4\right)$ and $\mathbf{S}f\in\mathcal{H}_2^{3/2+\varepsilon}\left(\mathbb{R}^3;\mathbb{C}^4\right)$. It suffices to show these inclusions for $t\geq 1$. Observe that

$$\begin{aligned} &\left\|\zeta\left(\tfrac{|x|}{R}\right)\mathbf{P}_\pm e^{-iH_0t}g\right\|_{L^2(\mathbb{R}^3)} = \left\|\mathcal{F}\left(\zeta\left(\tfrac{|x|}{R}\right)\mathbf{P}_\pm e^{-iH_0t}g\right)\right\|_{L^2(\mathbb{R}^3)} \\ &= (2\pi)^{-3/2}\left\|R^3\int_{\mathbb{R}^3}(\mathcal{F}\zeta)(R(p-q))\,e^{\mp i\sqrt{|q|^2+m^2}t}(\mathcal{F}\mathbf{P}_\pm g)(q)\,dq\right\|_{L^2(\mathbb{R}^3)}. \end{aligned} \tag{4.2}$$

For $g\in\mathcal{S}$, integrating by parts, we have

$$\int_{\mathbb{R}^3}(\mathcal{F}\zeta)(R(p-q))\,e^{\mp i\sqrt{|q|^2+m^2}t}(\mathcal{F}\mathbf{P}_\pm g)(q)\,dq$$
$$= \mp\frac{i}{t}\int_0^{\infty}\int_{\mathbb{S}^2} e^{\mp i\sqrt{|q|^2+m^2}t}\frac{q\cdot\nabla}{|q|}\left((\mathcal{F}\zeta)(R(p-q))\,|q|\sqrt{q^2+m^2}\,(\mathcal{F}\mathbf{P}_\pm g)(q)\right)d|q|\,d\omega,$$

with $q = |q|\,\omega$, and then, by Young's inequality,

$$\left\| \int_{\mathbb{R}^3} (\mathcal{F}\zeta)\,(R\,(p-q))\, e^{\mp i\sqrt{|q|^2+m^2}t}\,(\mathcal{F}\mathbf{P}_{\pm} g)\,(q)\,dq \right\|_{L^2(\mathbb{R}^3)} \leq C\frac{1}{t}\,\|\mathcal{F}g\|_{\mathcal{H}^2_{3/2+\varepsilon}}. \tag{4.3}$$

Moreover, by continuity, (4.3) holds for any $g \in \mathcal{H}_2^{3/2+\varepsilon}$. Hence, it follows from (4.2) that

$$\left\|\zeta\left(\frac{|x|}{R}\right)e^{-iH_0t}g\right\|_{L^2(\mathbb{R}^3)} \leq C\frac{1}{t}\,\|\mathcal{F}g\|_{\mathcal{H}^2_{3/2+\varepsilon}}. \tag{4.4}$$

Therefore, using (4.4) with $g = \mathbf{S}f$ and $g = f$, we conclude that the functions $t \to \left\|\zeta\left(\frac{|x|}{R}\right)e^{-iH_0t}\mathbf{S}f\right\|_{L^2(\mathbb{R}^3)}$ and $t \to \left\|\zeta\left(\frac{|x|}{R}\right)e^{-iH_0t}f\right\|_{L^2(\mathbb{R}^3)}$ belong to $L^2(\mathbb{R})$. Then, the assumptions of Proposition 4.1 are satisfied and (4.1) is valid.

Let us prove now the first assertion of Theorem 1.1. From the unitarity of the scattering operator $\mathbf{S}$, since $\mathcal{F}\mathbf{S}\mathcal{F}^{-1}$ commutes with $\frac{\sqrt{|p|^2+m^2}}{|p|}$ and $P_+(p) + P_-(p) = I$, we get

$$\sum_{\sigma=\pm}\left(\frac{\sqrt{|p|^2+m^2}}{|p|}f_\sigma(p), f_\sigma(p)\right)_{L^2(\mathbb{R}^3;\mathbb{C}^4)} = \sum_{\sigma=\pm}\left(\frac{\sqrt{|p|^2+m^2}}{|p|}(\mathbf{S}f)_\sigma(p), (\mathbf{S}f)_\sigma(p)\right)_{L^2(\mathbb{R}^3;\mathbb{C}^4)},$$

where $f_\pm(p) = P_\pm(p)\hat{f}(p)$ and $(\mathbf{S}f)_\pm(p) = P_\pm(p)\widehat{\mathbf{S}f}(p)$. Then, applying Theorem 3.2 to the R.H.S. of (4.1) we have

$$\begin{aligned}
\delta\mathcal{T}(f) = {} & i\int_{\mathbb{R}^3}\left\langle(\mathbf{S}f)_+(p), \tfrac{\sqrt{|p|^2+m^2}}{2|p|^2}(\mathbf{S}f)_+(p) + \tfrac{\sqrt{|p|^2+m^2}}{2|p|^2}p\cdot\nabla(\mathbf{S}f)_+(p) + \tfrac{1}{2|p|^2}p\cdot\nabla\left(\sqrt{|p|^2+m^2}\,(\mathbf{S}f)_+(p)\right)\right\rangle dp \\
& -i\int_{\mathbb{R}^3}\left\langle(\mathbf{S}f)_-(p), \tfrac{\sqrt{|p|^2+m^2}}{2|p|^2}(\mathbf{S}f)_-(p) + \tfrac{\sqrt{|p|^2+m^2}}{2|p|^2}p\cdot\nabla(\mathbf{S}f)_-(p) + \tfrac{1}{2|p|^2}p\cdot\nabla\left(\sqrt{|p|^2+m^2}\,(\mathbf{S}f)_-(p)\right)\right\rangle dp \\
& -i\int_{\mathbb{R}^3}\left\langle f_+(p), \tfrac{\sqrt{|p|^2+m^2}}{2|p|^2}f_+(p) + \tfrac{\sqrt{|p|^2+m^2}}{2|p|^2}p\cdot\nabla(f_+(p)) + \tfrac{1}{2|p|^2}p\cdot\nabla\left(\sqrt{|p|^2+m^2}f_+(p)\right)\right\rangle dp \\
& +i\int_{\mathbb{R}^3}\left\langle f_-(p), \tfrac{\sqrt{|p|^2+m^2}}{2|p|^2}f_-(p) + \tfrac{\sqrt{|p|^2+m^2}}{2|p|^2}p\cdot\nabla(f_-(p)) + \tfrac{1}{2|p|^2}p\cdot\nabla\left(\sqrt{|p|^2+m^2}f_-(p)\right)\right\rangle dp.
\end{aligned} \tag{4.5}$$

Noting that

$$\frac{\sqrt{|p|^2+m^2}}{2|p|^2}p\cdot\nabla f = \frac{1}{2|p|^2}p\cdot\nabla\left(\sqrt{|p|^2+m^2}f(p)\right) - \frac{1}{2\sqrt{|p|^2+m^2}}f,$$

we get

$$\begin{aligned}
& \frac{\sqrt{|p|^2+m^2}}{2|p|^2}f + \frac{\sqrt{|p|^2+m^2}}{2|p|^2}p\cdot\nabla f + \frac{1}{2|p|^2}(p\cdot\nabla)\left(\sqrt{|p|^2+m^2}f(p)\right) \\
& = \tfrac{1}{2}\left(\frac{2}{|p|^2}p\cdot\nabla\left(\sqrt{|p|^2+m^2}f(p)\right) + \frac{\sqrt{|p|^2+m^2}}{|p|^2}f\right) - \frac{1}{2\sqrt{|p|^2+m^2}}f \\
& = \frac{\sqrt{|p|^2+m^2}}{2}\left(2\frac{p\cdot\nabla}{|p|^2}f + \frac{1}{|p|^2}f\right) + \frac{1}{2\sqrt{|p|^2+m^2}}f,
\end{aligned} \tag{4.6}$$

Also, recalling (1.4) we have

$$\mathcal{F}\mathbf{A}_0 = \frac{i}{2}\left(2\frac{p\cdot\nabla}{|p|^2} + \frac{1}{|p|^2}\right)\mathcal{F}, \tag{4.7}$$

Therefore, using (4.6) and (4.7) in (4.5) we get (1.6).

We now prove the second affirmation of Theorem 1.1. Let us find the decomposition of the operator $\mathbf{T}$ in the spectral representation of the operator H_0. Passing to the spherical coordinate system in the integrals in the R.H.S. of (4.5) and making the change $E=\sqrt{r^2+m^2}$ in the first and third terms and $E=-\sqrt{r^2+m^2}$ in the other two terms we get

$$\begin{aligned} &\delta\mathcal{T}(f)= \\ &= i\int\limits_{(-\infty,-m)\cup(m,\infty)}\int\limits_{\mathbb{S}^2}\left\langle S(E)\Gamma_0(E)f,\tfrac{E}{2(E^2-m^2)}S(E)\Gamma_0(E)f+\tfrac{1}{2E}S(E)\Gamma_0(E)f+\partial_E\left(S(E)\Gamma_0(E)f\right)\right\rangle d\omega dE \\ &\quad -i\int\limits_{(-\infty,-m)\cup(m,\infty)}\int\limits_{\mathbb{S}^2}\left\langle\Gamma_0(E)f,\tfrac{E}{2(E^2-m^2)}\Gamma_0(E)f+\tfrac{1}{2E}\Gamma_0(E)f+\partial_E\Gamma_0(E)f\right\rangle d\omega dE. \end{aligned} \tag{4.8}$$

Suppose that the scattering matrix $S(E)$ is continuously differentiable with respect to E on some open set $O\subset(-\infty,-m)\cup(m,+\infty)\setminus\sigma_p(H)$ and $f\in\Phi(O)$, where $\Phi(O)$ is defined by (1.5). Then, from the unitarity of the scattering matrix $S(E)$ it follows that

$$\delta\mathcal{T}(f)=\int\limits_{(-\infty,-m)\cup(m,\infty)}\int\limits_{\mathbb{S}^2}\left\langle\Gamma_0(E)f,T(E)\Gamma_0(E)f\right\rangle d\omega dE, \tag{4.9}$$

where

$$T(E)=-iS(E)^*\frac{d}{dE}S(E).$$

The operators $T(E)$ are bounded in $L^2\left(\mathbb{S}^2\right)$. Relation (4.9) shows that the family $\{T(E)\}_{E\in(-\infty,-m)\cup(m,\infty)}$ defines a decomposition $\mathbf{T}$ in the spectral representation of H_0 for any $f\in\Phi(O)$. That is,

$$\left(\mathcal{F}_0\mathbf{T}f\right)(E,\omega)=T(E)\Gamma_0(E)f,\quad f\in\Phi(O).$$

Therefore, for any $f\in\Phi(O)$, the operator $\mathbf{T}$ is the Eisenbud-Wigner time delay operator.

Remark 4.2 We observe that the condition of strong differentiability of the scattering matrix may be relaxed (see page 485 of [3]). Actually, one can obtain (4.9) from (4.8) by only requesting strong continuity of $S(E)$, but in this case the operator $\frac{d}{dE}S(E)$ may be unbounded in $L^2\left(\mathbb{S}^2\right)$.

The proof of Theorem 1.1 is complete.

4.2 Proof of Theorem 1.5.

As in the case of the Schrödinger operator ([4]), the proof of Theorem 1.5 consists in showing that the assumptions of Theorem 1.5 imply the ones of Theorem 1.1. That is, we need to prove that for $\mathbf{V}$ satisfying Condition 1.4 and for $f\in\mathcal{D}_\tau$, $\tau>2$, the function $t\to\left\|\left(W_--e^{itH}e^{-itH_0}\right)f\right\|_{L^2(\mathbb{R}^3)}$ belongs to $L^1((-\infty,0])$, $t\to\left\|\left(W_+-e^{itH}e^{-itH_0}\right)\mathbf{S}f\right\|_{L^2(\mathbb{R}^3)}$ belongs to $L^1([0,\infty))$ and $f,\mathbf{S}f\in\mathcal{H}_2^{3/2+\varepsilon}\left(\mathbb{R}^3;\mathbb{C}^4\right)$, and, moreover, the scattering matrix $S(E)$ is strongly differentiable on some open set $O\subset(-\infty,-m)\cup(m,+\infty)\setminus\sigma_p(H)$, containing the support of ψ_f, given by the definition of $\mathcal{D}_\tau$. First of all, from the definition of the set $\mathcal{D}_\tau$ it follows that $f\in\mathcal{H}_2^{3/2+\varepsilon}\left(\mathbb{R}^3;\mathbb{C}^4\right)$. Next we note that the proofs of Lemmas 2 to 9 of [4] remain true if we consider the Dirac operator instead of the Schrödinger operator. We only make two remarks. Firstly, instead of relations (11) and (12) in Lemma 4 of [4], one has

$$[H,x_k]=-i\alpha_k$$

and

$$\left[H,|x|^m\right]=m\,|x|^{m-1}\sum_{j=1}^{3}\left(-i\alpha_j\right),$$

respectively. Then, the rest of the proof of Lemma 4, in the case of the Dirac operator, is similar, taking $\varepsilon=0$ in all of the formulas. Secondly, in Lemmas 5 to 8, in our case, we need that $\varphi\in C_0^\infty(\mathbb{R})$ be equal to 0 in some neighborhood of 0. By using Lemmas 1-9 of [4], we see that Corollary of Proposition 2 of [4] also holds in the case of the Dirac equation. Thus, in the way analogous to Proposition 3 of [4] we attain the inclusions $\left\|\left(W_--e^{itH}e^{-itH_0}\right)f\right\|_{L^2(\mathbb{R}^3)}\in L^1((-\infty,0])$ and $\left\|\left(W_+-e^{itH}e^{-itH_0}\right)\mathbf{S}f\right\|_{L^2(\mathbb{R}^3)}\in L^1([0,\infty))$. Observing that in the case of the Dirac operator $[x^2,H_0]=2i\alpha\cdot x$,

$$[x^2,U_t^*]=-i\left(\int\limits_0^{-t}U_{t+\tau}^*[x^2,H_0]U_\tau d\tau\right)=2i\left(\int\limits_0^{-t}U_{t+\tau}^*\left(-i\alpha\cdot x\right)U_\tau d\tau\right)=t^2U_t^*-2tU_t^*\left(\alpha\cdot x\right)$$

and

$$[x^2, U_t^0] = i\left(\int_0^{-t}\left(U_\tau^0\right)^*[x^2, H_0]U_{\tau+t}^0 d\tau\right) = -2i\left(\int_0^{-t}\left(U_\tau^0\right)^*(-i\alpha\cdot x)\,U_{\tau+t}^0 d\tau\right) = -t^2U_t^0 + 2t\,(\alpha\cdot x)\,U_t^0,$$

where $U_t^0 := e^{-itH_0}$ and $U_t := e^{-itH}$, and proceeding similarly to the proof of Proposition 4 of [4] we obtain $\mathbf{S}f \in \mathcal{D}_2 \subset \mathcal{H}_2^{3/2+\varepsilon}\left(\mathbb{R}^3;\mathbb{C}^4\right)$. Finally, the strong differentiability of the scattering matrix $S(E)$ on $(-\infty,-m)\cup(m,+\infty)\setminus\sigma_p(H)$ can be obtained from the stationary formula (2.7) in the way analogous to the case of the Schrödinger operator (see Theorem 3.5 of [18]) by using (2.2), (2.3) and the result about the differentiability of the resolvent for the Schrödinger operator given in Lemma 3.4 of [18]. Theorem 1.5 is proved.

4.3 Proof of Theorem 1.7.

Recall that for an operator A of trace class $\mathrm{Det}\,(I+A)$ is the determinant of $I+A$ ([38], [41]). Suppose that the potential $\mathbf{V}$ satisfies Condition 1.6. Then, it follows from Theorem 4.5 of [40] that the operator $S(E)-I$ is of trace class, the SSF $\xi(E;H,H_0)$ exists and it is related to the scattering matrix $S(E)$ by the Birman-Krein formula

$$\mathrm{Det}\,S(E) = e^{-2\pi i\xi(E;H,H_0)}. \tag{4.10}$$

Observe now that Condition 1.6 implies Condition 2.1. Moreover, under Condition 1.6 there are no eigenvalues embedded in the absolutely continuous spectrum of H (see Remark 2.2). Hence, (2.2) and (2.3) hold for any $E\in(-\infty,-m)\cup(m,\infty)$. Then, using (2.7), (2.2), (2.3) and the result about the differentiability of the resolvent for the Schrödinger operator given in Lemma 3.4 of [18], similarly to the case of the Schrödinger operator (see Proposition 1.9, page 300 of [41]), we prove that under Condition 1.6 $S(E)$ is continuously differentiable in the trace class. Therefore, as the scattering matrix is unitary, the operator $S(E)^*\frac{d}{dE}S(E)$ is of trace class and the following relation is satisfied

$$(\mathrm{Det}\,S(E))^{-1}\frac{d}{dE}\mathrm{Det}\,S(E) = \mathrm{Tr}\left(S(E)^*\frac{d}{dE}S(E)\right). \tag{4.11}$$

For the convenience of the reader we include the simple proof of (4.11). Let $\{f_n\}$ be an orthonormal basis of $L^2\left(\mathbb{S}^2;\mathbb{C}^4\right)$. We consider the square matrix $\{(S(E)f_n,f_m)\}$, where $n,m\le N$. Here $(\cdot,\cdot)$ denotes the scalar product of $L^2\left(\mathbb{S}^2;\mathbb{C}^4\right)$. By the definition of the determinant

$$\mathrm{Det}\,S(E) = \lim_{N\to\infty}\det\left(\{(S(E)f_n,f_m)\}\right). \tag{4.12}$$

Since $S(E)-I$ as of trace class, the limit in the R.H.S. of the last expression exists. Moreover, using Jacobi's formula we get

$$\frac{d}{dE}\det\left(\{(S(E)f_n,f_m)\}\right) = \det\left(\{(S(E)f_n,f_m)\}\right)\mathrm{Tr}\left(\{(S(E)f_n,f_m)\}^{-1}\left\{\left(\frac{d}{dE}S(E)f_n,f_m\right)\right\}\right).$$

Taking the basis $\{f_n\}$ that consists of eigenvalues of $S(E)$ we get

$$\frac{d}{dE}\det\left(\{(S(E)f_n,f_m)\}\right) = \det\left(\{(S(E)f_n,f_m)\}\right)\mathrm{Tr}\left(\{\lambda_{n,m}\}^{-1}\left\{\left(\frac{d}{dE}S(E)f_n,f_m\right)\right\}\right),$$

where $\lambda_{n,n}$ is the n-th eigenvalue of $S(E)$ and $\lambda_{n,m}=0$, for $n\neq m$. Thus,

$$\begin{aligned}\frac{d}{dE}\det\left(\{(S(E)f_n,f_m)\}\right) &= \det\left(\{(S(E)f_n,f_m)\}\right)\sum_{n=1}^{N}\left(\{\lambda_{n,n}\}^{-1}\frac{d}{dE}S(E)f_n,f_n\right) = \\ &= \det\left(\{(S(E)f_n,f_m)\}\right)\sum_{n=1}^{N}\left((S(E))^*\frac{d}{dE}S(E)f_n,f_n\right),\end{aligned}$$

where we used that as $S(E)$ is unitary $\lambda_{n,n}^{-1}=\lambda_{n,n}^*$. Taking the limit, as $N\to\infty$, we arrive to

$$\lim_{N\to\infty}\frac{d}{dE}\det\left(\{(S(E)f_n,f_m)\}\right) = (\mathrm{Det}\,S(E))\,\mathrm{Tr}\left(S(E)^*\frac{d}{dE}S(E)\right). \tag{4.13}$$

By (4.13)

$$\frac{d}{dE}\mathrm{Det}\,(S(E)) = (\mathrm{Det}\,S(E))\,\mathrm{Tr}\left(S(E)^*\frac{d}{dE}S(E)\right), \tag{4.14}$$

with the derivative in the left-hand side in distribution sense, but as $S(E)^*\frac{d}{dE}S(E)$ is continuous in the trace class, (4.14) holds with $\frac{d}{dE}\mathrm{Det}\,(S(E))$ in classical sense, what proves (4.11). On the other hand, similarly to Theorem 1.20, page 351 of [41] we show that $\xi(E;H,H_0)$ is continuous for $E\in(-\infty,-m)\cup(m,\infty)$. Hence, differentiating (4.10) and using (4.11) we see that $\xi(E;H,H_0)$ is differentiable on $(-\infty,-m)\cup(m,\infty)$ and, furthermore, we attain (1.8). Theorem 1.7 is proved.

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